Problem Set 3

Due on Tuesday, Oct 15

1. Let $f : 2^U \to \mathbb{R}_+$ be a monotone submodular function, and let F be its multilinear extension. Show that F is convex along any line of direction $\mathbf{e}_i - \mathbf{e}_j$ for unequal $i, j \in U$. ($\mathbf{e}_i$ is the vectors that is all 0 except the i-th coordinate being 1.)

2. (Williamson-Shmoys Exercise 4.6) This exercise introduces a deterministic rounding technique called *pipage rounding*. Let $G = (A, B, E)$ be a bipartite graph; that is, each edge $(i, j) \in E$ has $i \in A$ and $j \in B$. Assume that $|A| \leq |B|$ and that we are given nonnegative costs $c_{ij} \geq 0$ for each edge $(i, j) \in E$. A *complete matching* of A is a subset of edges $M \subseteq E$ such that each vertex in A has exactly one edge of M incident on it, and each vertex in B has at most one edge of M incident on it. We wish to find a minimum-cost complete matching. We can formulate an integer program for this problem in which we have an integer variable $x_{ij} \in \{0, 1\}$ for each edge $(i, j) \in E$, where $x_{ij} = 1$ if (i, j) is in the matching, and 0 otherwise. Then the integer program is as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} & \\
\text{subject to} \quad & \sum_{j\in B:(i,j)\in E} x_{ij} = 1, & \forall i \in A, \\
& \sum_{i\in A:(i,j)\in E} x_{ij} \leq 1, & \forall j \in B, \\
& x_{ij} \in \{0, 1\}, & \forall (i, j) \in E.
\end{aligned}$$

Consider the linear programming relaxation of the integer program in which we replace the integer constraints $x_{ij} \in \{0, 1\}$ with $x_{ij} \geq 0$ for all $(i, j) \in E$.

Show that given any fractional solution to the linear programming relaxation, it is possible to find in polynomial time an integer solution that costs no more than the fractional solution. (Hint: Given a set of fractional variables, find a way to modify their values repeatedly such that the solution stays feasible, the overall cost does not increase, and at least one additional fractional variable becomes 0 or 1.) Conclude that there is a polynomial-time algorithm for finding a minimum-cost complete matching.

3. Consider the following min-cost coverage problem. We are given a monotone submodular function $f : 2^V \to \mathbb{N}$ that is *integer-valued*; we are also given a target q, $0 \leq q \leq f(V)$. We are interested in finding $S \subseteq V$ as small as possible such that $f(S) \geq q$. That is, we wish to solve $S^* = \text{argmin}_S |S|$ s.t. $f(S) \geq q$. Consider the greedy algorithm from problem 3(b) in Problem Set 1. Let $S_0, S_1, \cdots$ be the sequence of sets picked by the greedy algorithm, and

let ℓ be the smallest index such that $f(S_\ell) \geq q$. Show that

$$\ell \leq \left(1 + \ln \max_{v \in V} f(\{v\})\right) \mathrm{OPT},$$

where $\mathrm{OPT} = \min_S |S|$ s.t. $f(S) \geq q$.