Problem Set 2

Due on Tuesday, Oct 8

For all the following problems, you may assume that any set function is normalized ($v(\emptyset) = 0$) and monotone ($\forall S \subseteq T$, $v(S) \leq v(T)$).

1. Show that the class of submodular functions is closed under nonnegative linear combinations. That is, given submodular functions $f_1, \cdots, f_n : 2^[m] \to \mathbb{R}_+$, and nonnegative reals $a_1, \ldots, a_n$, the function $f = \sum_i a_i f_i$ is still submodular.

2. We saw in class that, given a universe U and integer $k > 0$, $\mathcal{I} = \{S \subseteq U \mid |S| \leq k\}$ is the set of independent sets of a matroid, called the uniform k matroid on U.

   Consider the following generalization. Let $(P_1, \cdots, P_n)$ be a partition of U. That is, $P_1, \cdots, P_n \subseteq U$, $P_i \cap P_j = \emptyset$ for any $i \neq j$, and $\bigcup_i P_i = U$. Let $k_1, \cdots, k_n$ be nonnegative integers. A set $S \subseteq U$ is in $\mathcal{I}$ if and only if for $i = 1, \ldots, n$, $|S \cap P_i| \leq k_i$. Show that $(U, \mathcal{I})$ constitutes a matroid. This matroid is known as a *partition matroid.*

3. Many combinatorial optimization problems take the following form: given a matroid $(U, \mathcal{I})$ and weights $w : U \to \mathbb{R}$ on its elements, return for $S \in \mathcal{I}$ that maximizes $w(S) := \sum_{e \in S} w(e)$. Max weight spanning tree can be seen as an example, by constructing a matroid on a given undirected graph $G = (V, E)$ as follows: Take the edge set E as the universe, and let $\mathcal{I}$ be the set of all subsets of E that contain no cycle. One can show that $(E, \mathcal{I})$ is a matroid (convince yourselves), known as the *graphical matroid.*

   Show that the following greedy algorithm gives an optimal solution to the the problem of finding the max weight S: Initialize S as empty set; then while there exists an element $i \notin S$ with $S \cup \{i\} \in \mathcal{I}$ and $w(i) \geq 0$, find such an element with the maximum weight, and add it to S.

   (For many of you this is a review question. It is instructive to solve it without referring to your old notes.)

4. The problem of *submodular function maxmization subject to matroid constraints* takes as input a matroid $(U, \mathcal{I})$ and a submodular function $v : 2^U \to \mathbb{R}_+$ (accessed via value oracles), and seeks to find $S \in \mathcal{I}$ that maximizes $v(S)$. Show that, if we have a polynomial-time algorithm that gives an α-approximation for this problem, then we also have a polynomial-time algorithm that gives an α-approximation for the submodlar welfare problem. (That is, in a combinatorial auction where all the valuations are submodular, find allocation $S_1, \cdots, S_n$ that maximizes $\sum_i v_i(S_i)$.

5. Given a matroid $(U, \mathcal{I})$ and weights $w : U \to \mathbb{R}_+$, we may define the following set function $r : 2^U \to \mathbb{R}_+$, known as a *weighted rank function*: for any $S \subseteq U$, $r(S) := \max_{T \subseteq S, T \in \mathcal{I}} \sum_{e \in T} w(e)$. Show that r is gross substitute.

6. A valuation over a set L of items is said to be *OXS* if it can be described as follows: there is a bipartite graph $G = (L, R, E)$, with left vertex set L, right vertex set R and edge

set $E \subseteq L \times R$, and nonnegative edge weights $w : E \to \mathbb{R}_+$, such that for any $S \subseteq L$, $v(L)$ is the maximum weight of a matching that uses only vertices in S. That is, among all matchings containing only edges incident to vertices in S (and not to $L \setminus S$), the one with maximum weight gives the value of $v(S)$. Show that an OXS valuation is gross substitute.