Problem Set 1

Due on Tuesday, September 24

For all the following problems, you may assume that any valuation function is normalized ($v(\emptyset) = 0$) and monotone ($\forall S \subseteq T$, $v(S) \le v(T)$).

1. Given a profile of valuation functions, there can be more than one Walrasian equilibrium. Suppose $(S_1, \cdots, S_n, p_1, \dots, p_m)$ and $(S'_1, \cdots, S'_n, p'_1, \dots, p'_m)$ are two Walrasian equilibria. Show that $(S_1, \cdots, S_n, p'_1, \dots, p'_m)$ is still a Walrasian equilibrium. (This is sometimes known as the Second Welfare Theorem.)

2. In class we saw that the social welfare is maximized at a Walrasian Equilibrium. Recall that in a Walrasian equilibrium, no bidder would like to add or drop any item at the given prices. We may consider relaxing this condition: an allocation $S_1, \cdots, S_n$ and prices $p_1, \dots, p_m$ is said to be a PS1 equilibrium if:

   (a) no bidder would like to add any item: $\forall i, \forall T \supseteq S_i$, $v_i(S_i) - \sum_{j \in S_i} p_j \ge v_i(T) - \sum_{j \in T} p_j$;

   (b) any unallocated item has price 0: $\forall j \in M \setminus (\cup_i S_i)$, $p_j = 0$.

   (c) no bidder has negative utility: $\forall i, v_i(S_i) - \sum_{j \in S_i} p_j \ge 0$.

   Show that in any PS1 equilibrium, the social welfare is at least half of the optimal welfare. Formally, if $(S_1, \cdots, S_n, p_1, \dots, p_m)$ is a PS1 equilibrium, then for any allocation $S^*_1, \cdots, S^*_n$, $\sum_i v_i(S_i) \ge \frac{1}{2} \sum_i v_i(S^*_i)$.

3. Recall that a valuation function v is said to be *submodular* if it has decreasing marginal returns. That is, $\forall j, \forall S \subseteq T$, $v(\{j\} \cup S) - v(S) \ge v(\{j\} \cup T) - v(T)$.

   (a) With n bidders, each with a submodular valuation, consider the following way of finding an allocation: initialize $S_1 = \cdots = S_n = \emptyset$; then for each item $j = 1, 2, \dots, m$, give the item to the bidder whose current marginal value for j is the highest. Formally, let $i^* \in \arg\max_{i \in [n]} v_i(\{j\} \cup S_i) - v_i(S_i)$, update $S_{i^*} \leftarrow S_{i^*} \cup \{j\}$.

   (Here we assume the bidders' values are publicly accessible, and we do not need to worry about incentives.)

   Show that, when this procedure ends, the allocation $(S_1, \cdots, S_n)$ gives a 2-approximation to the optimal social welfare. That is, for any other allocation $S'_1, \cdots, S'_n$, we have $\sum_i v_i(S_i) \ge \frac{1}{2} \sum_i v_i(S'_i)$.

   (Hint: There are multiple ways to show this. One way is to make use of Problem 2. Can you come up with item prices to support $S_1, \cdots, S_n$ as a PS1 equilibrium?)

   (b) (Bonus) Given a bidder with a submodular valuation function v, suppose we would like to know, among all bundles of size k, which one is valued the most, but, instead of asking the bidder directly this question, we can only query her the values of specific bundles (such a query is called a *value query*). We may employ the following greedy algorithm: initialize S to be $\emptyset$; while $|S| < k$, find item j with the maximum marginal value $v(\{j\} \cup S) - v(S)$, add j to S, repeat.

Let S^G be the set returned by the greedy algorithm. Show that this is a $(1 - 1/e)$-approximation. That is, $\forall S$ such that $|S| = k$, $v(S^A) \geq (1 - \frac{1}{e})v(S)$.

(This problem may feel fairly challenging. Discussion is welcome and encouraged.)