Problem Set 4

Due date: Nov 30, 2017 in class

1. We have seen in class that the VCG mechanism is DSIC and maximizes the social welfare. However, the VCG mechanism uses as a subroutine a welfare maximization procedure, which is often a computationally hard task. Traditionally, computer scientists design approximation algorithms for hard problems, but if used in the VCG mechanism, approximation algorithms may not give rise to "approximate incentive copmatibility".

   We consider a generalization of single-item auctions where welfare optimization could be computationally hard. Let N be the set of bidders, and each bidder i has a private value v_i for being served. Let's assume each v_i is nonnegative. A set $S \subseteq N$ is said to be *feasible* if all the bidders in S could be simultaneously served. Let $\mathcal{F}$ be the set of all feasible sets. For example, in a single item auction, $\mathcal{F}$ is the set of all singleton sets. Now the welfare maximization problem $\max_{S \in \mathcal{F}} \sum_{i \in S} v_i$ could be NP-hard. (Convince yourself of this.) Let $\mathcal{A}$ be a polynomial time approximation algorithm that outputs, for every value profile $(v_1, \ldots, v_n)$, a set $S \in \mathcal{F}$ whose welfare is at least α fraction of the optimal. For each bidder i and value profile $\boldsymbol{v} = (v_1, \ldots, v_n)$, let $x_i(\boldsymbol{v})$ be the allocation rule of $\mathcal{A}$: $x_i(\boldsymbol{v}) = 1$ if i is in the set output by $\mathcal{A}$, and 0 otherwise. (Potentially we allow randomized allocation as well, in which case $x_i(\boldsymbol{v})$ is the probability with which bidder i is in the set of bidders served output by $\mathcal{A}$.)

   We know that, if we were to design an incentive compatible mechanisms, the key is to find an allocation rule that is monotone in each v_i. (Myerson's characterization easily extends to this setting. Convince yourself of this.) However, $\mathcal{A}$ may not be monotone. We address this by a assuming a Bayesian prior and relaxing the solution concept to Bayesian Incentive Compability: suppose the values are drawn independently from distributions $F_1, \cdots, F_n$, respecitively. Assume we know these distributions, and even have the ability to compute in polynomial time the interim allocation of $\mathcal{A}$. That is, if we denote by $x_i(v_i, \boldsymbol{v}_{-i})$ the allocation for bidder i given by $\mathcal{A}$, we assume we could compute precisely $x_i(v_i) := \mathbf{E}_{\boldsymbol{v}_{-i}}[x_i(v_i, \boldsymbol{v}_{-i})]$.[1]

   Show that one can compute a BIC mechanism which guarantees, in expectation, at least α fraction of the optimal welfare. More formally, show that there is a polynomial-time algorithm $\tilde{\mathcal{A}} : \prod_i T_i \to \mathcal{F}$ such that $\mathbf{E}_{v_1 \sim F_1, v_n \sim F_n}[\sum_{i \in \tilde{\mathcal{A}}(v_1, \ldots, v_n)} v_i] \geq \alpha\, \mathbf{E}_{v_1 \sim F_1, \ldots, v_n \sim F_n}[\max_{S \in \mathcal{F}} \sum_{i \in S} v_i]$, and for each $i \in N$, $\mathbf{Pr}_{\boldsymbol{v}_{-i}}[i \in \tilde{\mathcal{A}}(v_i, \boldsymbol{v}_{-i})]$ is monotone in v_i.

   For this problem, it suffices to describe the allocation rule of the BIC mechanism. (So you need not give the payment rule, which is immediate from the payment identity.) You also need not give precise runtime analysis, as long as the algorithm clearly runs in polynomial time.

   (**Hint:** Consider an analog with the ironing procedure.)

[1] In general this is not a realistic assumption; usually by sampling one can only get a close estimate of the interim allocation. We make this assumption to simplify the problem.

2. In this problem we explore a particular way of implementing the allocation in a single item auction, namely, by way of a *token-passing* procedure. In this procedure, initially the auctioneer holds the token. Then bidder 1 reveals her type t_1. According to a prespecified probability $\pi(\perp, t_1)$, the token is passed to bidder 1, where $\perp$ denotes the auctioneer. Then bidder 2 reveals her type t_2. If the token is still in the auctioneer's hands, then with prespecified probability $\pi(\perp, t_2)$, the token is passed to bidder 2; otherwise with prespecified probability $\pi(t_1, t_2)$, the token is passed from bidder 1 to bidder 2. The process continues: when bidder i shows up and reveals her type t_i; if currently bidder j with type t_j holds the item, the token is passed to bidder i with a prespecified probability $\pi(t_j, t_i)$, where t_j could be $\perp$ if the auctioneer has the token. After all bidders have been considered in this way, whoever has the token in the end will be allocated the item; if the auctioneer still holds the token, no bidder gets allocated.

   (a) Show that the set of all interim feasible allocations implementable by such a token-passing procedure could be described by the feasible set subject to a polynomial number of linear constraints (where by polynomial we mean polynomial in the length of the input, $\sum_i |T_i|$). You are allowed to introduce a polynomial number of auxiliary variables.

   (**Hint:** An obvious attempt would be to let the π's be some auxiliary variables and hope that the interim allocation rules can be expressed by them in a linear manner. If this fails, you may try to come up with other variables that describe the token-passing procedure in a linear way.)

   (b) Show that any ranking mechanism can be implemented by a token-passing procedure.

   (c) Deduce from the first two parts of the problem (and our characterization of interim feasible mechanisms) that all interim feasible mechanisms are implementable by a token-passing procedure, and that the polytope in the solution to part (a) can be used for optimization problems over all interim feasible mechanisms.

3. A *simultaneous first price item auction* is a way to run a combinatorial auction. Each bidder simultaneously places m bids on the m items, and then each item goes to the bidder that placed the highest bid on it. Each bidder then pays her bids on all the items she won.

   Show that any pure Nash equilibrium in a simultaneous first price auction constitutes a Walrasian equilibrium. (In this Walrasian equilibrium, the price of each item is the price paid by the winner for that item in the equilibrium of the simultaneous item auction, and the allocation is the same as in this equilibrium.)