Problem Set 3

Due date: Nov 9, 2017 in class

1. Describe the VCG allocation and payments for selling three items $\{a, b, c\}$ to three unit-demand bidders $\{1, 2, 3\}$, whose valuations are $v_1(a) = 2, v_1(b) = 5, v_1(c) = 7, v_2(a) = 8, v_2(b) = 6, v_2(c) = 8, v_3(a) = 0, v_3(b) = 10, v_3(c) = 5$.

2. Describe the allocation and payment rules (in the value space) of the revenue optimal auction for selling a single item to two bidders, whose values are drawn independently from $U[5, 10]$ and $U[5, 15]$, respectively, where $U[a, b]$ denotes the uniform distribution on the interval $[a, b]$.

3. Compute the bidding function at the symmetric Bayesian Nash equilibrium in an all pay auction, where each of n bidders has her private value drawn i.i.d. from a distribution F. Recall that in the all pay auction, the highest bidder wins, but everyone pays her own bid. In particular, what is the bidding function when F is the uniform distribution on $[0, 1]$?

4. Let us remove the assumption of value independence in the design of a revenue optimal auction. Suppose that the profile of the bidders' private values is drawn from a joint distribution D, which is known to the auctioneer.

   (a) We would like to design a DSIC mechanism that is ex post individually rational, i.e., for any realization of $\boldsymbol{v}$, bidder i's expected utility is nonnegative (when truthful bidding). Show that the optimal revenue extractable by such a mechanism is computable by a linear program.

   (b) We now experiment with the idea of relaxing the ex post IR constraint to that of *interim individual rationality*, that is, conditioning on any value of any bidder, her expected utility (when truthfully bidding) is nonnegative. Let's consider a specific example. Suppose there are two bidders whose values (v_1, v_2) are drawn according to the distribution: $\mathbf{Pr}[v_1 = 1, v_2 = 1] = 0.2$, $\mathbf{Pr}[1, 2] = 0.3$, $\mathbf{Pr}[2, 1] = 0.3$, $\mathbf{Pr}[2, 2] = 0.2$. Consider the following modification of the second price auction: we first run a second price auction (breaking ties, say, uniformly at random at equal bids), and then, (in addition to the second price auction payments), for each bidder, if her opponent bids 1, charge the bidder 1.8, and if her opponent bids 2, *pay* the bidder 1.2. (Note that in the second case we pay the bidder, i.e., the payment is negative.)

   Show that the mechanism is DSIC and interim IR, and calculate its revenue. Compare that with the maximum social, i.e., $\mathbf{E}[\max(v_1, v_2)]$.

   (c) (Bonus) Generalizing from the last question, can you come up with a family of auctions that extract large revenue (that is equal to the maximum welfare) for a family of distributions? State your result as general as you can.