Problem Set 2

Due date: Oct 31, 2017 in class

We consider a special case of combinatorial auctions called multi-unit auctions. In a multi-unit auction, all the m items are identical, and therefore the valuation function of each buyer is simply a non-decreasing function $v_i : [m] \to \mathbb{R}_+$. Bidder i's value for the j-th item given that she has already acquired $j-1$ items, $v_i(j) - v_i(j-1)$, is called her *marginal value* for the j-th item. If her marginal value never increases with j, we say her valuation exhibits *decreasing marginal values.*[1]

1. (a) Show that, when all bidders' valuations have decreasing marginal values, there is always a Walrasian equilibrium.

   (b) Show that, whenever a bidder has a valuation that violates the decreasing marginal values property, there exist other bidders with valuations having decreasing marginal values, such that when all these bidders are present in an auction, there exists no Walrasian equilibrium.

2. Describe the VCG prices for a multi-unit auction.

3. When bidders' valuations have decreasing marginal values, an ascending auction can be used to find a price that supports a Walrasian equilibrium: a publicly visible price p starts from 0 and keeps rising with all bidders instantaneously reporting her current demand, that is, $\arg\max_j v_i(j) - pj$, the number of items that would maximize a bidder's utility if she were to purchase at the current price p for each item; obviously, as the price increases, the demand of each bidder decreases; the auction stops when the total demand is equal to m, and then everyone pays the final price times her final demand. (You may use this procedure as part of your solution to 1(a), but there you need to show why a Walrasian equilibrium is obtained.) One way to turn this into a direct revelation mechanism is for each bidder to report a valuation vector in $\mathbb{R}_+^m$ upfront, and then the demand at each price level will be faithfully calculated according to the reported valuations. Is this mechanism incentive compatible? If so, give a proof. If not, give an example in which a bidder has incentive to lie about her true values.

4. We modify the ascending auction of the previous problem. At every price p, denote by $d_i(p)$ the number of items demanded by bidder i, and $d_{-i}(p)$ the total demand of bidders other than that of i: $\sum_{i' \neq i} d_{i'}(p)$. At price p, if $\sum_i d_i(p) > m$ and $d_{-i}(p) < m$, we know that bidder i will be allocated at least $m - d_{-i}(p)$ items. This is informative, and the idea of our modification will be to track the prices along the trajectory instead of waiting for the last moment and use a uniform price for all items. Specifically, for each bidder i we do the following: as p increases, till the auction ends (that is, when the total demand equals m), at each price such that d_{-i} drops, if $d_{-i} < m$, then $m - d_{-i}$ items are "presold" to bidder i; the *newly* presold items will each be charged at the price at which they are presold. Show that

[1] For those of you we have learned the concept of *submodularity* elsewhere, this is just submodularity in the context of valuations for homogeneous goods.

(if bidders report demands truthfully) this procedure charges the VCG prices for each bidder. Deduce that the corresponding direct revelation mechanism is incentive compatible.

For this problem, for simplicity, you may assume that all the mn marginal values are distinct, so that no tie-breaking needs to be considered.