Exercises for CPSC 536F

Updated on Oct 12, 2017

1 Review for probability theory

1. Let X be a nonnegative random variable whose cumulative distribution function is F. Show $\mathbf{E}[X] = \int_0^\infty (1 - F(x))\,\mathrm{d}x$.
2. If X and Y are independent random variables, show $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.
3. (Linearity of Expectations) Show $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$ for arbitrary random varaibles X and Y (in particular, even when they are not independent).
4. A couple has two kids. Assume that a newborn is male and female with probability half each.
   (a) Given that the elder one is a boy, what is the probability that the younger one is a boy?
   (b) Given that one of the two kids is a boy, what is the probability that the other one is a boy?
   (c) Given that the elder one is a boy and that he was born on a Tuesday, what is the probability that the younger one is a boy?
   (d) Given that one of the kids is a boy and that he was born on a Tuesday, what is the probaiblity that other one is a boy?
5. (Markov inequality) If X is a nonnegative random variable, show $\mathbf{Pr}[X \geq \lambda\,\mathbf{E}[X]] \leq \frac{1}{\lambda}$ for any $\lambda \geq 1$. Give examples of distributions for which this is tight.
6. (Chebyshev inequality) For any random variable X such that $\mathrm{Var}[X] = \sigma^2$, show $\mathbf{Pr}[|X - \mathbf{E}[X]| \geq \lambda\sigma] \leq \frac{1}{\lambda}$, for any $\lambda \geq 1$. Give example distributions for which this is tight.

2 Game theory basic solution concepts

1. Show that the game of rock-paper-scissors has a unique Nash equilibrium.
2. Show, by constructing an example, that a game that has a dominant strategy equilibrium can still have a pure Nash where each player is playing a dominated strategy.

   (When a dominant strategy exists, a strategy that is not a dominant strategy is said to be a dominated strategy.)
3. Compute a Nash equilibrium for the two-player zero-sum game where the utility for the row player is given by the matrix

$$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix}.$$

4. Compute a Nash equilibrium for the following resource-sharing game: 2 players compete for the use of one unit of divisible resource. The action space of each player i is an amount of money w_i that she would like to pay for her use of the resource. Given action vector (w_1, w_2), the amount of resource allocated to player i is $x_i = \frac{w_i}{w_1+w_2}$. Each user i's then gets a utility $u_i(x_i) - w_i$, where $u_1(x) = x - \frac{x^2}{4}$ and $u_2(x) = \frac{21}{2}(x - \frac{x^2}{2})$.

   Hint: It may be easier to first derive conditions for an equilibrium before substituting the expressions of u_1 and u_2.

3 Linear programming basics

1. Prove the variant of Farkas's lemma we used in class: the system $\{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is feasible if and only if the system $\{\mathbf{y}^\top A \geq \mathbf{0}, \mathbf{y}^\top \mathbf{b} < 0, \mathbf{y} \geq 0\}$ is infeasible. (**Hint:** Use Farkas's lemma.)

2. For a linear program

$$\begin{aligned} \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

   its Lagrangian $L(\lambda)$ ($\lambda \geq \mathbf{0}$) is defined as

$$\max_{\mathbf{x}} c^\top \mathbf{x} + \sum_i \lambda_i (b_i - \sum_j A_{ij} x_j).$$

   Assume the LP is feasible.

   (a) Show that, for any $\lambda \geq \mathbf{0}$, $L(\lambda)$ upper bounds the value of the LP.

   (b) Show that $\min_{\lambda \geq \mathbf{0}} L(\lambda)$ is equal to the value of the LP.

3. Verify that the two programs giving the upper and lower bounds for the rower's utility in a two-player zero-sum game are indeed primal and dual for each other.

4 Online Learning Basics

1. Derive and prove regret bound for a reward maximizing online learning algorithm (we did the cost minimizing version in class).

2. Show that the doubling trick keeps the regret at the same order, sacrificing only a constant compared with the algorithm which knows the time horizon in the first place.

3. Show that one can use the Hedge algorithm to get an algorithm with $O(\sqrt{nT \log n})$ swap regret.

5 Mechanism Design Basics

1. Verify the claims we made in the proof of Arrow's impossibility theorem. That is, for $S \subseteq [n], a, b \in A$, let $x_S(a,b)$ be 1 if $aV(\vec{p})b \Leftrightarrow \forall i \in S, ap_ib; \forall i \notin S, bp_ia$. Let $x_S(a,b)$ be 0 otherwise. Then

   (a) $x_S(a,b) + x_{\bar{S}}(b,a) = 1$, $\forall S \subseteq [n], \forall a, b \in A$.

   (b) $x_S(a,b) \leq x_S(a,c)$, $x_S(c,a) \leq x_S(b,a)$, $\forall S \subseteq [n], \forall a, b, c \in A$. Hence we can denote by x_S all variables $x_S(a,b)$. Let's say a set S is *powerful* if $x_S = 1$.

   (c) If S is powerful, then any $T \supseteq S$ is also powerful.

   (d) If S and T are powerful, then $S \cap T \neq \emptyset$.

   (e) If S and T are powerful, then so is $S \cap T$.

   (f) If S^* is the minimum powerful set, then $|S^*| = 1$.

6 The VCG mechanisms

1. Show that the second price auction is the VCG mechanism for single item auctions.

2. Show that the Generalized Second Price (GSP) auction is not dominant strategy incentive compatible. Derive the VCG mechanism for position auctions.

7 Combinatorial Auctions

1. Show that welfare maximiation for a combinatorial auction is NP-hard. (Proposition 1.5 in Section 1.2 of the assigned reading on October 12 contains a solution.)

2. Give an example showing that item prices in a Walrasian equilibrium do not necessarily give the VCG prices.

8 Submodular functions

1. Show that the two definitions we gave for submodular functions are equivalent.

2. Given a monotone submodular function f, suppose we would like to know its maximum value on all subsets of size k, and are only allowed to make value queries (that is, the query that takes a subset S, and returns $f(S)$). Consider the following greedy algorithm: initiate S to $\emptyset$; while $|S| < k$, find item j with the maximum marginal value $v(\{j\} \cup S) - v(S)$, add j to S, repeat.

   Let S^G be the set returned by the greedy algorithm. Show that this is a $(1-1/e)$-approximation. That is, $\forall S$ such that $|S| = k$, $v(S^A) \geq (1 - \frac{1}{e})v(S)$.

9 Linearity of Expectaitons

1. (Birthday Paradox) Let F be the uniform distribution on $[n] = \{1, 2, \cdots, n\}$. Show that, among $\sqrt{n}$ i.i.d. random variables drawn according to F, the expected number of pairs of them taking the same value is at least 1. Argue further that with $\Omega(\sqrt{n})$ i.i.d. random variables drawn according to F, with high probability there exist a pair that take the same value. (**Hint:** For the second question, you may consider using the Chernoff bound.)

2. (Coupon Collector's Problem) A coupon collector would like to collect k different coupons to take advantage of a promotion. He recieves an envelope each day, which contains one of the k coupons that is drawn uniformly at random. Show that the expected number of days that pass before she collects all k coupons is on the order of $k \log k$.

Solutions

Section 1 Problem 4: $\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{13}{27}$.
Section 2 Problem 3: The row player plays $(\frac{3}{4}, \frac{1}{4})$ and the column player plays $(\frac{1}{2}, \frac{1}{2})$.
Section 2 Problem 4: At an equilibrium, the two players must be best responding to each other. Player i's utility when the bids are w_1, w_2 is

$$R_i(w_1, w_2) = u_i\left(\frac{w_i}{w_1 + w_2}\right) - w_i.$$

For w_i be a best response, we should have

$$\frac{\partial R_i}{\partial w_i} = u_i'\left(\frac{w_i}{w_1 + w_2}\right) \cdot \frac{w_{3-i}}{(w_1 + w_2)^2} - 1 = 0.$$

Writing the allocation $\frac{w_i}{w_1+w_2}$ as x_i, this gives

$$u_i'(x_i)(1 - x_i) = w_1 + w_2.$$

At an equilibrium (w_1^*, w_2^*), this should hold for both utility functions:

$$u_1'(x_1)(1 - x_1) = w_1^* + w_2^* = u_2'(1 - x_1)x_1.$$

Substituting the derivatives of u_1 and u_2, we have

$$\left(1 - \frac{x_1}{2}\right)(1 - x_1) = [1 - (1 - x_1)]x_1.$$

Solving this quadratic equation gives $x_1 = \frac{1}{4}$, and hence $x_2 = \frac{3}{4}$. It is easy to verify that this constitutes an equilibrium. In fact, since the utility functions are strictly concave, it is not hard to see that the equilibrium is unique.
Section 7 Problem 2: In fact in the vast majority of cases the two are not equivalent. As a simple example, let there be two identical items, and bidder 1's values for one and two items are 3 and 4, respectively, and bidder 2's values are 2 and 4. The VCG prices are 2 for bidder 1 and 1 for bidder 2. The Walrasian prices are 2 for both players. (Walrasian prices are by definition anonymous; that is, anyone who would like to pick any specific item needs to pay the same price. A remarkable feature of the VCG mechanism is that prices are not anonymous. Bidders often need to pay different amounts for the same items, because the externalities they exert on each other are not symmetric.)
Section 8 Problem 2: Vondrák's lecture notes contain a solution to this problem.