Problem Set 2

Due on Thursday, October 13, 2016

For all the following problems, you may assume that any valuation function is normalized ($v(\emptyset) = 0$) and monotone ($\forall S \subseteq T$, $v(S) \leq v(T)$).

1. Given a profile of valuation functions, there can be more than one Walrasian equilibrium. Suppose $(S_1, \cdots, S_n, p_1, \ldots, p_m)$ and $(S'_1, \cdots, S'_n, p'_1, \ldots, p'_m)$ are two Walrasian equilibria. Show that $(S_1, \cdots, S_n, p'_1, \ldots, p'_m)$ is still a Walrasian equilibrium.

2. In class we saw that the social welfare is maximized at a Walrasian Equilibrium. Recall that in a Walrasian equilibrium, no bidder would like to add or drop any item at the given prices. We may consider relaxing this condition: an allocation $S_1, \cdots, S_n$ and prices $p_1, \ldots, p_m$ is said to be a PS2 equilibrium if:

   (a) no bidder would like to add any item: $\forall i, \forall T \supseteq S_i$, $v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(T) - \sum_{j \in T} p_j$;

   (b) any unallocated item has price 0: $\forall j \in M \setminus (\cup_i S_i)$, $p_j = 0$.

   (c) no bidder has negative utility: $\forall i, v_i(S_i) - \sum_{j \in S_i} p_j \geq 0$.

   Show that in any PS2 equilibrium, the social welfare is at least half of the optimal welfare. Formally, if $(S_1, \cdots, S_n, p_1, \ldots, p_m)$ is a PS2 equilibrium, then for any allocation $S^*_1, \cdots, S^*_n$, $\sum_i v_i(S_i) \geq \frac{1}{2} \sum_i v_i(S^*_i)$.

3. A valuation function v is said to be *submodular* if it has decreasing marginal returns. That is, $\forall j, \forall S \subseteq T$, $v(\{j\} \cup S) - v(S) \geq v(\{j\} \cup T) - v(T)$.

   Given a bidder with a submodular valuation function v, suppose we would like to know, among all bundles of size k, which one is valued the most, but, instead of asking the bidder directly this question, we can only query her the values of specific bundles (this is called *value queries*). We may employ the following greedy algorithm: initiate S to $\emptyset$; while $|S| < k$, find item j with the maximum marginal value $v(\{j\} \cup S) - v(S)$, add j to S, repeat.

   Let S^G be the set returned by the greedy algorithm. Show that this is a $(1-1/e)$-approximation. That is, $\forall S$ such that $|S| = k$, $v(S^A) \geq (1 - \frac{1}{e})v(S)$.

4. Gross substitute valuations. Show that a gross substitute valuation is submodular.