Problem Set 1

Due on Thursday, September 29, 2016

1. Compute a Nash equilibrium for each of the following games:

   (a) The two-player zero-sum game where the row player's payoff matrix is

   $$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix}$$

   (b) A resource-sharing game in which 2 players compete for the use of one unit of divisible resource. The action space of each player i is an amount of money w_i that she would like to pay for her use of the resource. Given action vector (w_1, w_2), the amount of resource allocated to player i is $x_i = \frac{w_i}{w_1+w_2}$. Each user i's then gets a utility $u_i(x_i) - w_i$, where $u_1(x) = x - \frac{x^2}{4}$ and $u_2(x) = \frac{21}{2}(x - \frac{x^2}{2})$.
   Hint: It may be easier to first derive conditions for an equilibrium before substituting the expressions of u_1 and u_2.

2. Consider a two player game where each player has m actions to take and the reward functions are given by two $m \times m$ matrices A and B, respecitvely. Suppose each entry of each matrix is drawn i.i.d. uniformly from the interval $[0, 1]$. Show that the probability that the game has a pure Nash equilibrium is at least $1 - 1/e$ when m is large. You may use the fact $(1 - 1/m)^m \approx 1/e$ for large m.

   Hint: When calculating the probabilities, pay special attention when the random events are not indenepdent.

   Bonus: Show that the probability in question in fact tends to (i.e., is not only lower bounded by) $1 - 1/e$ when m tends to infinity.

3. Problem 1.6 from the AGT book.

   Hint: You may use the Chernoff bound or the following Hoeffding's bound, whichever you find more convenient:

   Let $Z_1, \cdots, Z_m$ be a sequence of i.i.d. random variables and let $\bar{Z} = \frac{1}{m}\sum_i Z_i$. Assume that $\mathbf{E}[\bar{Z}] = \mu$ and $\mathbf{Pr}[a \le Z_i \le b] = 1$ for every i. Then, for any $\epsilon > 0$,

   $$\mathbf{Pr}\left[\left|\frac{1}{m}\sum Z_i - \mu\right| > \epsilon\right] \le 2e^{-\frac{2m\epsilon^2}{(b-a)^2}}.$$