CPSC 536F: Algorithmic Game Theory
Lecture: Review of Myerson's mechanism (including ironing) using revenue curves

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This is a short note that summarizes the key steps of deriving Myerson's mechanism, including ironing, using revenue curves. For full details, the reader is referred to the textbook by Jason Hartline.

1 Setting and Notations

We have a single item to sell, and each bidder i's value v_i is drawn independently from a knwon distribution whose cumulative density function is F_i with derivative f_i.

We denote by $x_i(v_i, v_{-i})$ the allocation to bidder i when the bid/value profile is v_i and v_{-i}, and $p_i(v_i, v_{-i})$ the payment made by bidder i. We use $x_i(v_i)$ to denote the *interim* allocation when bidder i's value is v_i, i.e., $x_i(v_i) = \mathbf{E}_{v_{-i}}[x]_i(v_i, v_{-i})$, and similarly for $p_i(v_i)$.

2 Derivation

We derive Myerson (1981)'s optimal mechanism in six steps. Seeing (ironed) virtual values as the derivatives of the (ironed) revenue curve comes from Bulow and Roberts (1989), although the actual proof idea here comes from Alaei et al. (2013).

1. **Characterization of Bayesian incentive compatible mechanisms.** Every BIC mechanism has monotone allocation rule, i.e., $x_i(v_i)$ is nondecreasing with v_i. Moreover, the expected payment is determined by the allocation rule: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(s)\,\mathrm{d}s$.

   Note: The characterzation is really more about IC than about BIC. For example, any DSIC mechanism must have its allocation rule $x_i(v_i, v_{-i})$ monotone in v_i given any v_{-i}, and the payment $p_i(v_i, v_{-i})$ is also determined as $v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(s, v_{-i})\,\mathrm{d}s$.

2. **Decomposition into step functions.** Any monotone allocation rule is the convex decomposition of step functions. In other words, the function $x_i(v_i)$ can be written as the weighted sum of some step functions, and the weights are nonnegative and sum to 1. (In the continuous case, we have an integral instead of a sum.) We would like to determine the weights or density of these step functions in this decomposition. Let us try to determine the weight of the step function that jumps from 0 to 1 at v. Intuitively, the value v's allocation is $x_i'(v)\,\mathrm{d}v$

more than the value that is slightly below it; this difference should be the probability that this particular step function is used, and so the weight should be $x_i'(v)\,\mathrm{d}v$.

3. **Calculating revenue using posted prices.** By revenue equivalence (i.e., that payment is determined by the allocation rule), to implement any monotone allocation rule, it is equivalent to randomize over a set of allocation functions that are step functions, where the probability of running the step function that jumps from 0 to 1 at value v is $x_i'(v)\mathrm{d}v$. Such a step function is implemented by a posted price at v, and its expected revenue is $v(1-F_i(v))$. The expected revenue of any allocation rule x_i is therefore

$$\int_0^\infty [v(1-F(v))]x_i'(v)\,\mathrm{d}v. \tag{1}$$

Note that the integral over v here is not with respect to the density f_i. We can already do an integral by part at this point, and using the fact that $v(1-F(v))$ evaluates to 0 at both 0 and ∞, this integral is equal to

$$\int_0^\infty x(v)[vf_i(v)-(1-F_i(v))]\,\mathrm{d}v = \int_0^\infty x_i(v)\left[v-\frac{1-F_i(v)}{f_i(v)}\right]f_i(v)\,\mathrm{d}v.$$

The last step, extracting the factor $f_i(v)$ from the bracket, gives us the expression for *virtual surplus* with respect to the *virtual value*. This is the expression in Myerson's original proof. Note that here, by having the density function as the measure, it is as if we are taking expectation with respect to v drawn from its original distribution. This meaning was not there in (1). This change of meaning in the integral variable is crucial, and it is one of the motivations for us to move from the value space to quantile space.

4. **Passing to the quantile space.** As we have seen, the integral (1) is with respect to a distribution of step functions (or posted prices), given by $x'(v)\,\mathrm{d}v$, where the measure for v itself is uniform. We could as well carry over to a uniform distribution on the compact domain $[0,1]$ through the mapping $\psi_i(v)=1-F_i(v)$. $\psi_i(v)$ is called the *quantile* of the value v. Let $y_i:[0,1]\to[0,1]$ be the quantile allocation function, that is, $y_i(q)=x_i(\psi_i^{-1}(q))$. In the decomposition of the allocation rule, the step function jumping at v then has weight/density $-y_i'(q)\,\mathrm{d}q$ evaluated at $q=\psi_i(v)$.[1] Let $R_i(q)=q\cdot F_i^{-1}(1-q)$ be the revenue of the step function at (or, equivalently, the posted price of) $v=\psi^{-1}(q)$, the revenue (1) can be rewritten, in terms of quantiles, as

$$\int_0^1 R_i(q)(-y_i'(q))\,\mathrm{d}q = \int_0^1 R_i'(q)y_i(q)\,\mathrm{d}q, \tag{2}$$

where the equality again follows by integral by part. $R_i(q)$ is called the *revenue curve.*

Definition 1. A distribution F_i is said to be regular if its regular curve is concave.

Remark: Passing to the quantile space may seem strange at first. One of the advantages of this switch is that it facilitates a perspective change. Instead of thinking about $v(1-F(v))$,

[1] As a sanity check, $-y_i'(q)=-\frac{\mathrm{d}x_i(\psi_i(v))}{\mathrm{d}v}\cdot\frac{\mathrm{d}v}{\mathrm{d}\psi_i(v)}\Big|_{v=\psi_i^{-1}(q)}=x_i'(\psi_i^{-1}(q))$, agreeing with our calculation before.

the revenue of a certain posted price, $R(q)$ suggests the revenue of a selling strategy that sells with *ex ante* probability q: setting a price at $\psi_i^{-1}(q)$ is only one of such strategies. This perspective immediately leads to a more general definition of revenue curve and ironing itself.

5. **Generalization of revenue curves and Ironing.** Let $\tilde{R}_i(q)$ be the optimal revenue extractable from bidder i with an incentive compatible mechanism that sells with ex ante probability q. Then obviously $\tilde{R}_i(q) \geq R_i(q)$ for any $q \in [0,1]$. Furthermore, by step 2, any IC mechanism itself can be implemented by a distribution over posted prices. Therefore $\tilde{R}_i(q)$ is simply the concave hull of $R_i(q)$.[2]

   Note: For any q where $\tilde{R}_i(q) > R_i(q)$, the revenue of the posted price $\psi_i^{-1}(q)$ is less than the revenue of a distribution over two other posted prices, whose expected ex ante selling probability is just q.

   Now, the revenue of any BIC mechanism from bidder i is

$$\int_0^1 R_i'(q)y_i(q)\,\mathrm{d}q = \int_0^1 R_i(q)(-y_i'(q))\,\mathrm{d}q \leq \int_0^1 \tilde{R}_i(q)(-y_i'(q))\,\mathrm{d}q = \int_0^1 \tilde{R}_i'(q)y_i(q)\,\mathrm{d}q. \qquad (3)$$

6. **The optimal mechanism.** The optimal mechanism maximizes its revenue with respect to the RHS of (3), and in fact achieves it. By the inequality in (3), such a mechanism maximizes the revenue as well, with equality therein attained.

   Recall that $y_i(q)$ is the allocation of bidder i when her value is $v = \psi_i^{-1}(q)$. Therefore, in order to maximize $\int_0^1 \tilde{R}_i'(q)y_i(q)\,\mathrm{d}q$, the optimal mechanism solicits bids $v_1, \ldots, v_n$, and maps them to quantiles $\psi_1(v_1), \ldots, \psi_n(v_n)$, then observes the corresponding $\tilde{R}_1'(\psi_1(v_1)), \cdots, \tilde{R}_n'(\psi_n(v_n))$. If the maximum among these is above zero, then allocate the item to this bidder; otherwise, do not sell.

   Remark: The quantity $\tilde{R}_i'(q)$ is the *ironed virtual value* of bidder i's type that has quantile q. It is "ironed" because in any region (q_1, q_2) where $\tilde{R}_i$ is strictly greater than R_i, $\tilde{R}_i$ is a straight line and has all types in that region have the same ironed virtual value, and therefore, in the optimal mechanism, they are all treated the same. Equivalently, any posted price whose selling probability lies in (q_1, q_2) is used with probability 0 in the optimal mechanism. In other words, the allocation rule is flat on (q_1, q_2). This is also necessary for the equality in (3) to be attained.

References

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[2] Equivalently, the region enclosed by $\tilde{R}_i(q)$ with the q-axis is the convex hull of that enclosed by $R_i(q)$ and the q-axis.