CPSC 536F: Algorithmic Game Theory
Lecture: The Lookahead Auction and Second Price Auctions with Reserve Prices

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1 The Lookahead Auction

Ronen (2001) designed the following Lookahead auction for a single item auction where the bidders' values are drawn from a correlated distribution.

Definition 1. In the Lookahead auction, the highest bidder i^* is offered a take-it-or-leave-it price which is the optimal posted price for the conditional distribution $F^{\text{win}}_{v^* | v_{-i^*}}$, the distribution of v_{i^*} conditioning on all other bidders' values v_{-i^*} and the fact that v_{i^*} is no lower than any of them.

Proposition 1. *The lookahead auction is DSIC.*

Proof. The bidder with the highest value has no incentive to bid below the second highest bid, as he would immediately lose the auction. But all any bid above the second highest bid, the auction is the same. For any bidder i whose value is not the highest, bidding anything below v_{i^*} makes no difference, and bidding above v_{i^*} will lead to a price that is at least $v_{i^*} \geq v_i$, which bidder i will reject anyway. Therefore bidding truthfully is a dominant strategy for every biddder. □

Definition 2. An auction is *ex post individually rational* (IR) if, for any bidder i, at any value profile $v_1, \ldots, v_n$, the bidder receives a nonnegative utility, i.e., $v_i x_i(v_i, v_{-i}) \geq p_i(v_i, v_{-i})$. An auction is *interim IR* if any bidder i with any value v_i receives a nonnegative utility in expectation, i.e., $v_i \mathbf{E}_{v_{-i} \sim D_{v_{-i} \mid v_i}}[x_i(v_i, v_{-i})] \geq \mathbf{E}_{v_{-i} \sim D_{v_{-i} \mid v_i}}[p_i(v_i, v_{-i})]$, where $D_{v_{-i} \mid v_i}$ denotes the conditional distribution of v_{-i} given v_i.

Theorem 1 (Ronen, 2001)**.** *The Lookahead auction extracts at least half of the revenue of any DSIC, ex post IR mechanism.*

Proof. Let us condition the analysis on the event that bidder 1 is the highest bidder and that other bidders bid v_{-1}. The revenue of the optimal auction under this event consists of two parts: H, the expected revenue extracted from bidder 1, the highest bidder; and L, the revenue extracted from the rest of the bidders. Any DSIC mechanism conditioning on the event (that bidder 1 is highest, and the rest of the bids are $v_2, \ldots, v_n$) is a DSIC mechanism for bidder 1. By definition, the lookahead auction is revenue optimal among all such mechanisms. Hence its revenue is at least H. One way to run a DSIC auction for the highest bidder is simply to charge him the second highest bid. Bidder 1 for sure can take this price, but this in turn upper bounds L for any ex post IR mechanism. To see this more clearly, suppose v_2 is the second highest bid, let $x_2, \ldots, x_n$ be the fractions of the item given to the lower bidders, and $p_2, \ldots, p_n$ their expected payments. Then $x_i v_i \geq p_i$ for each $i \geq 2$. So $\sum_{i=2}^n p_i \leq \sum_{i=2}^n v_i x_i \leq v_2 \sum_{i=2}^n x_i \leq v_2$.) Therefore the lookahead auction's revenue is no less than both H and L, and hence gets at least half of the optimal revenue. □

Remark 1. In this proof we crucially used both DSIC and ex post IR of the benchmark optimal mechanism, as opposed to BIC or *interim individual rationality*. Interim IR may look like a slight relaxation of the stronger ex post IR, but for most correlated distribution, Crémer and McLean (1985) showed a somewhat counterintuitive auction that fully extracts the social surplus, meaning that the item is efficiencly allocated (to the bidder with the highest value), but all value created goes to the auctioneer in expectation, and all bidders end up with zero utility. This auction is beyond the scope of this course.

2 Second price auction with reserve prices

The lookahead auction is revenue optimal among all auctions that sell only to the highest bidders. For independent regular distributions, the lookahead auction takes on a particularly nice form.

Definition 3. For a bidder whose value is drawn from the distribution F, the *monopoly reserve* for this bidder is the posted price that extracts the most revenue, i.e., $\arg\max_p p(1 - F(p))$.

Definition 4. The *second price auction with lazy monopoly reserves* selects the highest bidder i^*, and offers her a take-it-or-leave-it price $\max\{r_{i^*}, v_2\}$, where r_{i^*} is the monopoly reserve price for i^*, and v_2 is the second highest bid.

Theorem 2. *When bidders' valuations are drawn independently from regular distributions, the lookahead auction is the second price auction with lazy monopoly reserve prices.*

Proof. Let i^* be the highest bidder, and v_2 be the second highest bid. We only need to see that the price we set for i^* in the lookahead auction is equal to $\max\{r_{i^*}, v_2\}$.

The distribution of v_{i^*}, conditioning on $v_{i^*} \geq v_2$ is just F_{i^*} truncated at v_2, i.e., $F_{i^*}^{\text{win}}(v) = \frac{F_{i^*}(v) - F(v_2)}{1 - F_{i^*}(v_2)}$, for any $v \geq v_2$, and $F_{i^*}^{\text{win}}(v) = 0$ for any $v < v_2$. Setting any price below v_2 would be obviously suboptimal, and the revenue of setting a price of $p \geq v_2$ is

$$p(1 - F_{i^*}^{\text{win}}(p)) = p \cdot \left(1 - \frac{F_{i^*}(p) - F_{i^*}(v_2)}{1 - F_{i^*}(v_2)}\right) = p(1 - F_{i^*}(p)) \cdot \frac{1}{1 - F_{i^*}(v_2)}.$$

In words, for prices above v_2, the conditional expected revenue is simply scaled up by a constant factor of $\frac{1}{1 - F_i(v_2)}$ from the unconditioned distribution. Therefore, for $v_2 < r_i$, the optimal price to set for the conditional distribution is still r_i; and for $v_2 \geq r_i$, by regularity, the expected revenue monotonically decreases as the price rises above v_2; therefore the optimal price to set is v_2 itself. To summarize, we have shown that the price we set for i^* in the lookahead auction is equal to $\max\{r_{i^*}, v_2\}$. □

Corollary 1 (Dhangwatnotai et al., 2010)**.** *For independent regular bidders, the second price auction with lazy monopoly reserves achives at least half of the optimal revenue.*

Remark 2. The reserves are called *lazy* here because they are applied at the very end of the auction. One could consider eliminating upfront all bidders who bid below their reserves. We will get a second price auction with *eager* monopoly reserves. Hartline and Roughgarden (2009) showed that the latter also gives a 2-approximation for independent regular bidders.

3 Auction with single samples

In the previous lecture, we have seen that, for a single bidder whose value is drawn from a regular distribution, posted an independent random sample from the same distribution as a take-it-or-leave-it price extracts at least half of the optimal revneue. Let us use the lookahead auction to generalize the idea to multiple bidders.

Proposition 2. *In the lookahead auction, if the highest bidder is offered a distribution of posted prices whose expected revenue is an α-approximation to the optimal revenue for the conditional distribution of her value ($F^{\text{win}}_{v_{i^*} \mid v_{-i^*}}$), then the resulting auction is a 2α-approximation to the optimal revenue.*

Lemma 3. *For a bidder with regular value distribution F, conditioning on her value is above v, then sampling an independent value r from her distribution, and posting $\max\{r, v\}$ as a take-it-or-leave-it price gives a 2-approximation to the optimal revenue extractable from this conditional distribution.*

Proof. As we have seen in the proof of Theorem 2, the revenue of any posted price above v for the conditional distribution is simply the revenue of that price for F scaled by a factor of $\frac{1}{1-F(v)}$. We have seen that for the unconditional distribution F, a randomly sampled price gives at least half of the optimal revenue. By the same argument, if we could draw a sample from the conditional distribution as a posted price, that would give us a 2-approximation as well. But with probability $F(v)$, the sample r is smaller than v, and we need to use v instead. We will argue that this is still good enough for a 2-approximation.

Let r^* be the monopoly reserve, and let $R(p) = p(1-F(p)) \cdot \frac{1}{1-F(v)}$ be the revenue of the posted price $p \geq v$ for the conditional distribution. For $v \geq r$, by regularity, $R(v) \geq R(r)$ for any $r \geq v$. So with probability $1 - F(v)$ we have a sample reserve from the conditional distribution, and with probability $F(v)$ we use v which is a better reserve than any from the conditional distribution. So this case is easy. For $v < r^*$, imagine if we could use a sample r from F and magically get a revenue of $r(1 - F(r)) \cdot \frac{1}{1-F(v)}$ for all r, then we inherit the approximation ratio from the unconditional case. However, for $r < v$, we use v instead of r. But by regularity, $v(1 - F(v)) \geq r(1 - F(r))$ for any $r < v < r^*$. Therefore our revenue is even better than the imagined scenario, and is still at least half of the optimal revenue. $\square$

Definition 5. The second price auction with single sample reserves offers the highest bidder i^* a take-it-or-leave-it price equal to $\max\{v_2, s_{i^*}\}$, where v_2 is the second highest bid, and s_{i^*} is an independent sample from the value distribution F_{i^*}.

Corollary 2 (Dhangwatnotai et al., 2010)**.** *The second price auction with single sample reserves gives a 4-approximation to the optimal revenue.*

Remark 3. Note that the second price auction with single sample reserves reduces our reliance on the value distributions to a single sample from each value distribution. This is an example of prior-independent auction design.

References

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