Final Exam

Due on Sunday, December 18th, 2016, at 10am

You may refer to any lecture notes, homework problems, or textbooks we have used this semester. But otherwise you are expected to solve the problems independently. As in the homework, partial credit will be given for partially successful attempts.

1. (2 points each)

   (a) In the rock-paper-scissors game, suppose we double the payoff for the pair (rock, scissors) for both players, solve the Nash equilibrium of the new game. Formally, solve the Nash equilibrium of the two-player zero-sum game with the payoff matrix

   $$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}.$$

   (b) Describe the VCG allocation and payments for selling three items $\{a, b, c\}$ to three unit-demand bidders $\{1, 2, 3\}$, whose valuations are $v_1(a) = 2, v_1(b) = 5, v_1(c) = 7, v_2(a) = 8, v_2(b) = 6, v_2(c) = 8, v_3(a) = 0, v_3(b) = 10, v_3(c) = 5$.

   (c) Describe the allocation and payment rules (in the value space) of the revenue optimal auction for selling a single item to two bidders, whose values are drawn independently from $U[5, 10]$ and $U[5, 15]$, respectively, where $U[a, b]$ denotes the uniform distribution on the interval $[a, b]$.

2. (3 points) Consider the following procedure for selling m items to n bidders with monotone submodular valuations: initialize $S_1 = \cdots = S_n = \emptyset$; for each item j, let i^* be the bidder with the highest marginal value for j given S_i, i.e., $i^* = \arg\max_i v_i(S_i \cup \{j\}) - v_i(S_i)$, breaking ties arbitrarily, and add j to S_{i^*}: $S_{i^*} \leftarrow S_{i^*} \cup \{j\}$. Show that, when the algorithm finishes, $(S_1, \cdots, S_n)$ is an allocation that 2-approximates the optimal social welfare. That is,

   $$\sum_i v_i(S_i) \geq \frac{1}{2} \sum_i v_i(S_i^*),$$

   for any allocation S_i^*. (For this problem, do not consider incentive issues. It is not hard to see that the algorithm is not incentive compatible. Assume the value functions $v_1, \ldots, v_n$ are available to the algorithm.)

3. (3 points) Consider the following computational problem of finding approximately optimal reserve prices. In a single item auction, we are given m value profiles from n bidders: $(v_1^1, \ldots, v_n^1), \ldots, (v_1^m, \ldots, v_n^m)$. We would like to compute optimal reserve prices so that, when the second price auction with (eager) reserve prices is run on these value profiles (for m

times), the total revenue is maximized. More formally, we would like to find reserve prices $\vec{r^*} = (r_1^*, \dots, r_n^*)$, such that, the revenue

$$\sum_{j=1}^{m} \mathrm{Rev}^{\vec{r^*}}(v_1^j, \dots, v_n^j) \tag{1}$$

is maximized, where $\mathrm{Rev}^{\vec{r^*}}(\vec{v})$ denotes the revenue of the second price auction with eager reserves $\vec{r^*}$ on the value profile $\vec{v}$. Let OPT be the optimal value of (1).

This problem is in general hard. Given a polynomial time algorithm and prove that its output $\vec{r}$ guarantees a 2-approximation to OPT.

(**Hint:** Think about the Lookahead auction.)

4. Similar to the simultaneous first price auction, in a simultaneous second price auction, n bidders simultaneously put bids on each of m items, and then each item is allocated to the highest bidder on it, and the winner of each item is charged the second highest bid on that item.

   (a) (2 points) Convert the outcome of the algorithm in Problem 2 to a pure Nash equilibrium of the simultaneous second price auction. That is, construct bidding strategies for all players in a simultaneous second price auction, such that they consitute a pure Nash equilibrium and the resulting allocation is the one produced by the algorithm in Problem 2. Briefly explain why the strategies form an equilibrium.

   (b) (3 points) In class we saw that the second price auction has strange equilibria when bidders overbid, which result in unbounded PoA. In this problem we consider eliminating such bluffing and analyze the remaining equilibria. In a bidding profile $(b_1, \cdots, b_n)$ in a simultaneous second price auction, we say that the bidders are not overbidding if, in the resulting allocation $(S_1, \cdots, S_n)$, for every bidder i, $v_i(S_i) \geq b_i(S_i) = \sum_{j \in S_i} b_{ij}$.

   Show that any pure Nash equilibrium of a simultaneous second price auction in which no bidder overbids gives an allocation whose social welfare is at least half of the optimal social welfare. That is, the pure PoA of the simulnateous second price auction is at most 2.

   (**Hint:** You may find a homework problem useful, although the problem is not much harder without resorting to that.)