

# Conditional Equilibrium Outcomes via Ascending Price Processes with Applications to Combinatorial Auctions with Item Bidding\*

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## Abstract

A Walrasian equilibrium in an economy with non-identical indivisible items exists only for small classes of players' valuations (mostly “gross substitutes” valuations), and may not generally exist even with decreasing marginal values. This paper studies a relaxed notion, “conditional equilibrium”, that requires individual rationality and “outward stability”, i.e., a player will not want to *add* items to her allocation, at given prices. While a Walrasian equilibrium outcome is unconditionally stable, a conditional equilibrium outcome is stable if players cannot choose to drop only *some* of their allocated items.

With decreasing marginal valuations, conditional equilibrium outcomes exhibit three appealing properties: (1) An approximate version of the first welfare theorem, namely that the social welfare in any conditional equilibrium is at least half of the maximal welfare; (2) A conditional equilibrium outcome can always be obtained via a natural ascending-price process; and (3) The second welfare theorem holds: any welfare maximizing allocation is supported by a conditional equilibrium. In particular, each of the last two properties independently implies that a conditional equilibrium always exists with decreasing marginal valuations (whereas a Walrasian equilibrium generally does not exist for this common valuation class). Additional strategic foundation is provided via a strong connection to Nash equilibria of combinatorial auctions with item bidding, which enables us to strengthen results of Bhawalkar and Roughgarden (2011) for this auction game.

Given these appealing properties we ask what is a maximal valuation class that ensures the existence of a conditional equilibrium and includes unit-demand valuations. Our main technical results provide upper and lower bounds on such a class. The lower bound shows that there exists such a class that is significantly larger than gross-substitutes, and that even allows for some (limited) mixture of substitutes and complements. For three items or less our bounds are tight, implying that we completely identify the unique such class. The existence proofs are constructive, and use a “flexible-ascent” auction that is based on algorithms previously suggested for “fractionally subadditive” valuations. This auction is slightly different from standard ascending auctions, as players may also decrease prices of obtained items in every iteration, as long as their overall price strictly increases.

JEL Classification Numbers: C70, D44, D82

Keywords: Walrasian Equilibrium, Ascending auctions

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\*We thank David Easley and Ron Holzman for many helpful comments and advice. This work was supported by a grant from the USA-Israel Bi-national Science Foundation.

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# 1 Introduction

**Background and Motivation.** The classic notion of a Walrasian equilibrium (WE) focuses attention on equilibrium outcomes, from which no individual (player) wishes to deviate. In such an outcome (i.e., allocation plus item prices), each player is being allocated her most preferred bundle of items, under given prices. While early economic theory studies divisible economies, for which a WE always exists whenever valuations are convex, there is by now also a rich literature on economies with *indivisible* items. A valuation in this context is a function from subsets of items to the reals, i.e., the player assigns a real value for every bundle (subset of items). This representation can capture rich structures of substitutabilities and complementarities among the items.

The theoretical foundations of a Walrasian equilibrium reveal several attractive properties. For example, the first welfare theorem shows that any allocation that is supported by a Walrasian equilibrium has optimal social welfare, and the second welfare theorem shows the converse, i.e., any allocation with optimal social welfare can be coupled with appropriate prices to form a Walrasian equilibrium. Furthermore, a Walrasian equilibrium can be obtained by a natural ascending-price iterative process, similar to the one-item English auction, in which players repeatedly report demands at current prices, and prices of over-demanded items increase. Kelso and Crawford (1982) identify a class of “gross-substitutes” valuations that ensures the existence of a WE with indivisible items, and Gul and Stacchetti (2000) and Ausubel (2006) prove that, with gross-substitutes valuations, the ascending-price process always terminates in a WE.

However, with indivisible items, the *existence* of a Walrasian equilibrium is rather limited. Gul and Stacchetti (1999) show that gross substitutes is the *maximal* class of valuations that contains unit-demand valuations and ensures existence of a WE. This is a problem since this class is extremely small. Lehmann, Lehmann and Nisan (2006) describe a hierarchy of valuation classes, in which gross-substitutes is strictly contained in the class of valuations with decreasing marginal values, that in turn is strictly contained in a class of “fractionally subadditive” valuations, that is a strict subset of all subadditive valuations.<sup>1</sup> Moreover, Lehmann et al. (2006) show that *gross-substitutes valuations have zero measure amongst all valuations with decreasing marginal values*. In this paper we ask what can be said when valuations are not gross substitutes.

To answer this question, one needs to first understand the source of the failure to have a WE when valuations are not gross-substitutes, which is well demonstrated by examining the course of the ascending-price auction. The key property that ensures termination in a WE with gross-substitutes valuations is that demanded items whose price does not increase continue to be demanded. Thus every item is demanded by at least one player in every iteration. This implies that at the end, all items are allocated, and every player receives a bundle that she demands, and this is exactly a WE. Without gross substitutability, players sometime drop previously demanded items, even if

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<sup>1</sup>The name “fractionally subadditive” is due to Feige (2006), the same class is termed XOS in Lehmann et al. (2006).

their price did not increase. Thus, some items may be left unallocated, while their price is strictly positive. *This happens even with decreasing marginal values.*

**Conditional Equilibrium.** In light of this, and assuming that some more regulation is possible, a natural modification to the ascending-price process suggests itself: restrict players’ demands, so that a player will not be allowed to drop items that she previously demanded, if their price did not increase. Thus, in each step of the auction, the player will take her *conditional* demand, given items she already obtained. The end outcome of this process can be viewed as a relaxation of a Walrasian equilibrium, that satisfies two properties: (1) it is individually rational (whenever valuations are marginally decreasing), i.e., no player pays more than her resulting value, and (2) it is “outward-stable”, i.e., no player wishes to “take” items allocated to others, paying their listed price. Motivated by this observation, our first step in this paper is to define the notion of a “conditional equilibrium” (CE), requiring these two properties to be satisfied. We then embark on a systematic investigation of the implications of this notion.

With marginally decreasing valuations, we show that a conditional equilibrium outcome guarantees all three appealing properties of a Walrasian equilibrium, only for a significantly larger class of valuations. First, it admits an approximate version of the first welfare theorem: the social welfare of any conditional equilibrium outcome is at least half of the optimal social welfare. In fact this theorem holds regardless of the valuation class being considered, exactly as the original first welfare theorem. Second, as explained above, a conditional equilibrium outcome can be obtained via a natural ascending-price process. Third, the second welfare theorem holds: any allocation with optimal social welfare is supported by a conditional equilibrium.

In an attempt to fully understand how general is this state of affairs, we ask what kind of valuations always ensure the *existence* of a CE. More accurately, we seek a maximal class of valuations which includes unit-demand valuations<sup>2</sup> and ensures the existence of a CE. Any class of valuations meeting these criteria will be denoted in this paper by  $V^{CE}$  and will be said to *satisfy the MAXCE requirements*. From the above discussion it follows that there exists a class satisfying the MAXCE requirements that includes all marginally decreasing valuations, but perhaps there exists such a class which is significantly larger?

Answering this question turns out to be a challenging task, and our main results give partial answers. We show a class of valuations that upper bounds (contains) any  $V^{CE}$ , and another class that lower bounds (is contained in) at least one  $V^{CE}$ . For three items (or less), these two bounds coincide, resulting in a complete answer and proving that  $V^{CE}$  is unique. The general characterization question — and even the uniqueness question, for more than three items — remains open for future research. Fortunately, even with our partial results, we already know that  $V^{CE}$  can be

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<sup>2</sup>To avoid pathologies, it is necessary to require the inclusion of a basic “anchor” valuation class, large enough to include all valuations in which the bidder desires one particular item in  $\Omega$  and has no marginal value for other elements of  $\Omega$ . Gul and Stacchetti (1999) argue that it is reasonable to require the inclusion of unit-demand valuations as the anchor. Our result holds for other choices of such anchor classes as well, for example additive valuations.

quite large: it can contain all “fractionally subadditive” valuations, along with some additional valuations that allow for a mixture of sub-additivity and super-additivity (i.e. a mixture of substitutabilities and complementarities is possible). Given our results, we believe that further study of this characterization question will be interesting, conceptually as well as technically.

Without decreasing marginal values, even if a CE exists, the above-mentioned ascending-price process will not necessarily reach it. Nevertheless, our existence proofs are constructive, and suggest a slightly different ascending-price process based on algorithms presented by Christodoulou, Kovács and Schapira (2008) and by Dobzinski, Nisan and Schapira (2005). As in the more standard format suggested above, in each round a player adds her conditional demand to items previously taken by her and not demanded by others since then. However, instead of slightly increasing the price of each new item that she takes, the player is allowed to arbitrarily adjust (increase and/or decrease) prices of items in her possession, as long as her total price strictly increases. This gives the player more flexibility in “defending” her items, and yields a more robust auction format that leads to CE outcomes for larger valuation classes.

**Combinatorial Auctions with Item Bidding.** Our results yield some interesting connections to the strategic setting of combinatorial auctions with item bidding. In this setting,  $m$  different items are allocated to  $n$  players via  $m$  simultaneous single-item auctions. As in our setting, each player has a valuation function that assigns a real value to any possible set of items. The seller, however, requires the bidders to simply bid for each item independently and simultaneously. This simplifies the auction structure, but inserts various strategic issues. Having fixed the valuation functions of the players, the recent literature on this subject analyzes this setting as a normal-form game with complete information, and studies its Nash equilibria.

Hassidim, Kaplan, Mansour and Nisan (2011) assume that the seller uses first-price auctions, and show that the pure Nash equilibria of the resulting game exactly correspond to the Walrasian equilibria of the economy defined by the players’ valuation functions. In particular, existence of a pure Nash equilibrium is guaranteed if and only if valuations are gross-substitutes. The problem of limited existence of WE appears again. Christodoulou et al. (2008) and Bhawalkar and Roughgarden (2011) show that if we replace the first-price rule by a second-price rule, existence is guaranteed for a much larger class of valuations. The properties of the end outcome when using a second-price rule are thus weaker than that of a WE. But what are these properties?

We show a strong connection of the end outcome with a second-price rule to conditional equilibrium (which is indeed a relaxation of WE): there exists a pure Nash equilibrium with the second-price rule if and only if there exists a conditional equilibrium for the same tuple of valuations.<sup>3</sup> Moreover, the underlying transformation between the NE and the CE preserves the same allocation of items, hence also the same resulting social welfare.

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<sup>3</sup>Following Christodoulou et al. (2008) and Bhawalkar and Roughgarden (2011) we only look at Nash equilibria that satisfy “weak no-overbidding”, as explained in section 2.

Via this connection, our results for conditional equilibrium immediately yield two implications for combinatorial auctions with item bidding using the second-price rule. First, assuming valuations satisfy free disposal, any NE of this complete information game has social welfare of at least half the optimal welfare. This generalizes a main result of Bhawalkar and Roughgarden (2011), that shows the same statement, assuming that players' valuations are subadditive. We show that this assumption is not needed, and the claim holds even if players' values exhibit arbitrary complementarities. Second, our existence results for CE translate to existence of pure NE in the item bidding game. In particular, pure NE in this game are guaranteed to exist even if valuations exhibit certain patterns of complementarities, our upper and lower bounds on existence classes are relevant for this game, and the concluding open question gets additional motivation.

The remainder of this paper is organized as follows. In Section 2 we formally define the basic setup and prove some fundamental properties. Section 3 expands the results to the class of “fractionally subadditive” valuations, via a more flexible ascending-price auction. Section 4 studies the characterization of maximal valuation classes that ensure the existence of a CE, denoted as  $V^{CE}$ . It describes a class that upper bounds (contains) any such  $V^{CE}$ , and another class that lower bounds (is contained in) at least one  $V^{CE}$ . It also gives further evidence that our lower bound is probably too weak when allocating more than three items. The full answer is left open for future research. Section 5 concludes.

## 2 Conditional Equilibrium

We study a combinatorial auction setting with  $n$  players that value subsets of indivisible items from  $\Omega$ , the set of all items. Each player has a valuation function  $v_i : 2^\Omega \rightarrow \mathfrak{R}_+$ , that assigns a real value  $v_i(S_i)$  to any bundle  $S_i \subseteq \Omega$ . It is assumed that  $v_i(\emptyset) = 0$  and  $v_i(S) \leq v_i(T)$  for  $S \subseteq T \subseteq \Omega$ . For  $T, U \subseteq \Omega$  such that  $T \cap U = \emptyset$ , we use the notation  $v_i(T|U)$  to denote  $v_i(T \cup U) - v_i(U)$ . In other words,  $v_i(T|U)$  is the *marginal value* of  $T$  given that the player already has  $U$ . An *allocation*  $S = (S_1, \dots, S_n)$  is a partition of the items (player  $i$  is allocated the items in  $S_i$ ), i.e.,  $\cup_{i=1}^n S_i = \Omega$  and  $S_i \cap S_j = \emptyset$  for every distinct  $i, j \in \{1, \dots, n\}$ .

**Definition 1** (Conditional Equilibrium (CE)). An allocation  $S = (S_1, \dots, S_n)$  and a vector of item prices  $p = (p_x)_{x \in \Omega}$  form a *conditional equilibrium (CE)* for valuations  $v = (v_1, \dots, v_n)$  if, for all  $i = 1, \dots, n$ ,

1. (Individual Rationality)  $v_i(S_i) \geq p(S_i) = \sum_{x \in S_i} p_x$
2. (Outward Stability)  $\forall T \subseteq \Omega \setminus S_i, v_i(T|S_i) \leq p(T)$ .

We say that a tuple of valuations  $v = (v_1, \dots, v_n)$  *admits a CE* if there exists an allocation  $S = (S_1, \dots, S_n)$  and prices  $p = (p_x)_{x \in \Omega}$  that form a conditional equilibrium for  $v$ .

If  $(p, S)$  form a CE, the allocation  $S$  is “outward-stable” in the sense that no player wishes to *expand* her bundle of items at the given prices (assuming that utilities are quasi-linear). In contrast, a Walrasian equilibrium is unconditionally stable, in the sense that a player does not wish to add and/or remove items from the set of items allocated to her. The relaxation makes sense, for example, when the seller is able to require each buyer to either accept at least all items allocated to her, or none of the items. In a conditional equilibrium outcome, buyers will choose to accept the allocation and in particular will not wish to expand it. The following four important parallels between the two equilibrium notions serve as additional evidence to the interesting connection that exists between the two.

**The First Welfare Theorem.** The social welfare of allocation  $S$  is the sum of players’ values of their allocated bundles, i.e.,  $\sum_{i=1}^n v_i(S_i)$ . The first welfare theorem (specialized to quasi-linear preferences) implies that any allocation of a Walrasian equilibrium has the maximal social welfare among all possible allocations. A conditional equilibrium exhibits an approximate version of the first welfare theorem, as follows.

**Proposition 1.** *Fix a tuple  $v = (v_1, \dots, v_n)$  of valuations and let  $S^* = (S_1^*, \dots, S_n^*)$  be an allocation with maximal social welfare. If  $(S, p)$  is a conditional equilibrium for  $v$  then*

$$\sum_{i=1}^n v_i(S_i) \geq \frac{1}{2} \sum_{i=1}^n v_i(S_i^*).$$

*Proof.* For every player  $i$  we have  $v_i(S_i^* \setminus S_i | S_i) \leq p(S_i^* \setminus S_i)$ . Thus, we have

$$v_i(S_i^*) \leq v_i(S_i^* \cup S_i) \leq p(S_i^* \setminus S_i) + v_i(S_i) \leq p(S_i^*) + v_i(S_i).$$

Since  $S$  and  $S^*$  are allocations we have  $\sum_{i=1}^n p(S_i) = p(\Omega) = \sum_{i=1}^n p(S_i^*)$ . All of this implies

$$\sum_{i=1}^n v_i(S_i^*) \leq p(\Omega) + \sum_{i=1}^n v_i(S_i) = \sum_{i=1}^n p(S_i) + \sum_{i=1}^n v_i(S_i) \leq 2 \sum_{i=1}^n v_i(S_i),$$

where the last inequality follows since  $p(S_i) \leq v_i(S_i)$  by the individual rationality requirement of a CE. □

This analysis is tight as shown by the following example: let  $\Omega$  be  $\{a, b, c, d\}$ , and suppose there are two bidders with valuations  $v_1, v_2$  defined as follows. For any non-empty bundle  $S \subseteq \Omega$ ,  $v_1(S)$  is 1 unless  $\{a, c\} \subseteq S$ , in which case  $v_1(S)$  is 2; similarly,  $v_2(S)$  is 1 unless  $\{b, d\} \subseteq S$ , in which case  $v_2(S)$  is 2. Now let  $S_1 = \{a, b\}$ ,  $S_2 = \{c, d\}$ ; let  $p(b) = p(c) = 1$ ,  $p(a) = p(d) = 0$ . It is easy to check that this constitutes a CE with social welfare of 2, whereas the optimal social welfare is 4.

**With Decreasing Marginal Valuations: The Outcome of a Natural Ascending-Price Auction.** A natural way to obtain a Walrasian equilibrium is to perform an ascending price

process, where players report demands, and prices of over-demanded items increase (Gul and Stacchetti, 2000). This process terminates in a Walrasian equilibrium only when valuations are gross-substitutes, which is unfortunately a rather limited class of valuations. When valuations belong to a class that is broader than gross-substitutes, even if they are marginally decreasing, the ascending price process generally fails to terminate in a Walrasian equilibrium. In contrast, a conditional equilibrium can be obtained for any tuple of marginally decreasing valuations via a very similar ascending-price process. Formally, we say that a valuation  $v_i$  has decreasing marginal values if for any disjoint sets  $S_i, T_i \subseteq \Omega$  and any  $S'_i \subseteq S_i$ ,  $v_i(T_i|S'_i) \geq v_i(T_i|S_i)$ .<sup>4</sup> The following auction process finds a conditional equilibrium for any tuple of such valuations.

**Definition 2** (The Traditional-Ascent Auction). Fix some price increment parameter  $\Delta$ . Set initial item prices  $p_x$  ( $x \in \Omega$ ) to zero, and initial allocation  $S_1, \dots, S_n$  to be empty ( $S_i = \emptyset$ ). Then repeat:

1. Choose a player  $i$  with a non-empty conditional demand, i.e., such that every set  $T \in \operatorname{argmax}_{T \subseteq \Omega \setminus S_i} (v_i(T|S_i) - p(T))$  is non-empty. Fix a minimal such  $T$  with respect to containment.
  2. Increase the price: fix an arbitrary item  $x \in T$  and set  $p_x \leftarrow p_x + \Delta$ .
  3. Update tentative allocation:  $S_i \leftarrow S_i \cup T$  and  $S_j \leftarrow S_j \setminus T$  for any  $j \neq i$ .
- ... until all players have empty conditional demands.

**Proposition 2.** *When all valuations are marginally decreasing, with values that are multiples of  $\Delta$ , the Traditional-Ascent Auction terminates in a conditional equilibrium.*

*Proof.* Outward stability of the end outcome of the ascending auction is by definition. We show individual rationality, by arguing that the tentative allocation  $(S_1, \dots, S_n)$  is individually rational with respect to the prices  $p$  at the end of *any* iteration. In fact, we show that for any player  $i$  and any  $S \subseteq S_i$  we have  $v_i(S) \geq p(S)$ .

It suffices to prove the claim only for the player who changes her tentative allocation in the given iteration, as the claim will then follow by induction. Suppose player  $i$  has conditional demand  $T$  at the beginning of the iteration and tentative allocation  $S_i \setminus T$ , and let  $p'$  be the vector of item prices at the beginning of the iteration. First note that for any  $T' \subseteq T$ ,  $v_i(T'|S_i \setminus T') > p'(T')$ , since  $T$  is a minimal conditional demand, hence  $v_i(T|S_i \setminus T) - p'(T) > v_i(T \setminus T'|S_i \setminus T) - p'(T \setminus T')$ , which by rearranging gives the required inequality.

Now fix some  $S \subseteq S_i$ . Let  $S' = S \cap (S_i \setminus T)$  and  $T' = S \cap T$ . We have by induction  $v_i(S') \geq p'(S') = p(S')$ . We also have  $v_i(T'|S') \geq v_i(T'|S_i \setminus T') > p'(T')$ , where the first inequality follows since  $v_i$  has decreasing marginal values, and the second inequality follows from the previous paragraph. Thus  $v_i(T'|S') \geq p'(T') + \Delta \geq p(T')$ . As a result,  $v_i(S) = v_i(S') + v_i(T'|S') \geq p(S') + p(T') = p(S)$ , and the claim follows.  $\square$

<sup>4</sup>It is well-known that a valuation has decreasing marginal values if and only if it is submodular.

**With Decreasing Marginal Valuations: The Second Welfare Theorem.** As a converse of the first welfare theorem, the second welfare theorem states that, for gross-substitutes valuations, an allocation with maximal social welfare can always be supported by appropriate prices, to form a Walrasian equilibrium. The same statement is true for a conditional equilibrium, only for the significantly larger class of marginally decreasing valuations. We state the theorem in a somewhat more general way that will be useful for the sequel, using the following notion of “supporting prices”.

**Definition 3** (Dobzinski et al. (2005)). Given a valuation  $v_i$  and a subset  $S \subseteq \Omega$ , prices  $\{p_x\}_{x \in S}$  are *supporting prices* for  $v_i(S)$  if  $v_i(S) = \sum_{x \in S} p_x$ , and for every  $T \subseteq S$ ,  $v_i(T) \geq \sum_{x \in T} p_x$ .

We say that a valuation  $v$  admits supporting prices if for any  $S \subseteq \Omega$  there exist supporting prices for  $v(S)$ . Dobzinski et al. (2005) prove that there always exist supporting prices when the valuation is marginally decreasing; in fact they prove a more general result that we discuss in the sequel.<sup>5</sup> For completeness let us sketch a proof of this fact. Given a valuation function  $v$  and a subset of items  $S = \{x_1, x_2, \dots, x_{|S|}\}$ , set prices  $p_{x_\ell} = v(x_\ell | \{x_1, x_2, \dots, x_{\ell-1}\})$  for any  $1 \leq \ell \leq |S|$ . We immediately have that  $v_i(S) = \sum_{\ell=1}^{|S|} p_{x_\ell}$ . For every  $T \subseteq S$ , denote  $T = \{x_{j_1}, \dots, x_{j_{|T|}}\}$ , where  $j_i \leq j_{i+1}$  for every  $1 \leq i < |T|$ . Since  $v$  is marginally decreasing we have  $v(x_{j_i} | \{x_{j_1}, \dots, x_{j_{i-1}}\}) \geq v(x_{j_i} | \{x_1, \dots, x_{j_{i-1}}\}) = p_{x_{j_i}}$ , which implies  $v_i(T) = \sum_{i=1}^{|T|} v(x_{j_i} | \{x_{j_1}, \dots, x_{j_{i-1}}\}) \geq \sum_{i=1}^{|T|} p_{x_{j_i}}$ . Thus  $p$  are supporting prices for  $v(S)$ .

The second welfare theorem for conditional equilibrium holds whenever valuations admit supporting prices, as we next show.

**Proposition 3.** *Fix any tuple of valuations that admit supporting prices. Then for any welfare-maximizing allocation  $S$  there exist prices  $p$  such that  $(S, p)$  is a conditional equilibrium.*

*Proof.* Let  $(S_1, S_2, \dots, S_n)$  be any allocation that maximizes the social welfare for these valuations. For each  $S_i$ ,  $1 \leq i \leq n$ , there exists a supporting price vector  $p^i$  for  $v_i$ . For every player  $i$  and every item  $x \in S_i$ , define the price  $p(x)$  to be equal to  $p^i(x)$ . Individual rationality follows immediately from the definition of supporting prices. To prove outward stability, fix a player  $i$  and any subset  $T$  that is disjoint from  $S_i$ . We must prove that  $v_i(T | S_i) \leq p(T)$ . Denote  $R_j = T \cap S_j$ . The fact that  $(S_1, \dots, S_n)$  maximizes social welfare implies that  $v_i(S_i \cup T) + \sum_{j \neq i} v_j(S_j \setminus R_j) \leq \sum_{j=1}^n v_j(S_j)$ , hence  $v_i(T | S_i) \leq \sum_{j \neq i} v_j(R_j | S_j \setminus R_j) \leq \sum_{j \neq i} p(R_j) = p(T)$ , where the last inequality follows from the definition of supporting prices.  $\square$

**Corollary 1.** *Fix any tuple of marginally decreasing valuations. Then for any welfare-maximizing allocation  $S$  there exist prices  $p$  such that  $(S, p)$  is a conditional equilibrium.*

<sup>5</sup>Supporting prices are exactly the anti-core of the cooperative TU game  $v(\cdot)$ , see additional details in section 3 below. Thus, the claim that there always exist supporting prices when the valuation is marginally decreasing also follows from Shapley’s theorem that the marginal worth vectors belong to the anti-core of a submodular game.

**Strategic Foundation: the connection to Nash equilibria of Combinatorial Auctions with Item Bidding.** In a combinatorial auction with item bidding, goods are sold separately using  $|\Omega|$  single-item auctions held simultaneously. Viewing this as a normal form game with complete information (having fixed an arbitrary fixed tuple of valuations  $(v_1, \dots, v_n)$ ), several recent papers analyze its Nash equilibria. If the single-item auctions being used are first-price auctions, Hassidim et al. (2011) show that there exist pure Nash equilibria of this game if and only if there exist Walrasian equilibria for  $(v_1, \dots, v_n)$ . In particular, with first-price auctions, pure Nash equilibria are guaranteed to exist only if valuations are gross-substitutes. If the single-item auction being used is a second-price auction, Christodoulou et al. (2008) show that pure Nash equilibria exist for the much larger class of *fractionally subadditive* valuations (we give a detailed discussion on this class in section 3 below). Thus, conceptually, moving from first-price to second-price expands existence at the cost of weakening the properties of the end outcome from Walrasian equilibria to some weaker solution concept. It turns out that the correct solution concept to use for this case is a CE.

Before stating a formal claim, we need to be careful about one important detail. Formally, having fixed  $(v_1, \dots, v_n)$ , consider the following normal-form game (Second-Price Item Bidding (SPIB)): the strategy of player  $i$  is a vector of bids  $b_i^1, \dots, b_i^{|\Omega|}$ . Each item is then allocated to a player with a highest bid (and ties are decided arbitrarily), and the price of that item equals the second-highest bid for it. The resulting utility of player  $i$  is her value for the subset of items she receives, minus the sum of prices of these items.

Now, an important issue is that for this game, a Nash equilibrium in fact always exists: one player bids “infinity” on all items, while all other players bid zero on all items. Since we are using a second-price auction, the player that bids infinity receives all items, and pays zero. Obviously she is best responding. No other player can improve her utility, since, to win an item, she will have to overbid the “infinity” bids, resulting in a negative utility. Therefore all other players are best responding as well, and we have a Nash equilibrium.

To rule out these equilibria which are not so interesting, Christodoulou et al. (2008) and Bhawalkar and Roughgarden (2011) restrict attention to *Nash equilibria with no overbidding*, which are those Nash equilibria  $\vec{b}$  in which, for any player  $i$  and any subset of items  $S \subseteq \Omega$ ,  $v_i(S) \geq \sum_{j \in S} b_i^j$ . In other words, a player does not bid more than her value on some subset of items. We will consider here a weaker restriction, *Nash equilibria with weak no-overbidding*, which are those Nash equilibria  $\vec{b}$  in which for any player  $i$ ,  $v_i(S_i(\vec{b})) \geq \sum_{j \in S_i} b_i^j$ , where  $S_i(\vec{b})$  is the set of items that player  $i$  receives when the vector of bids is  $\vec{b}$ . In other words, player  $i$  can overbid on many subsets; the no-overbidding restriction involves only the specific set of items that player  $i$  receives. This relaxation of the original no-overbidding will become important as we will consider in the sequel non-subadditive valuations, for which the original no-overbidding restriction does not make any sense: it may be that the value of a set of several items is significantly larger than the sum

of values of a certain partition of this set to several subsets. The original no-overbidding condition may effectively limit the players to bid very low (relative to their actual value) on certain sets of items.

**Proposition 4.** *Fix any tuple of valuations  $(v_1, \dots, v_n)$ . There exists a pure Nash equilibrium with weak no-overbidding in the SPIB normal-form game for  $(v_1, \dots, v_n)$  if and only if there exists a CE for  $(v_1, \dots, v_n)$ . Moreover, the underlying transformation between the NE and the CE preserves the same allocation of items, hence also the same resulting social welfare.*

*Proof.* Suppose first that there exists a pure Nash equilibrium with weak no-overbidding  $\vec{b}$  for  $(v_1, \dots, v_n)$ . We construct a CE for  $(v_1, \dots, v_n)$  in the following way: the set of items  $S_i$  that player  $i$  receives in the CE is exactly the set of items she receives in the SPIB game with bid vector  $\vec{b}$ . The price of item  $j \in \Omega$  in the CE is  $p_j = \max_{i=1, \dots, n} b_i^j$ . By definition, for every player  $i$  and every  $j \in S_i$ ,  $b_i^j = p_j$ . Thus, by weak no-overbidding,  $v_i(S_i) \geq \sum_{j \in S_i} p_j$ , implying the individual rationality requirement. Since  $\vec{b}$  is a Nash equilibrium, for any player  $i$  and any  $T \subset \Omega \setminus S_i$ ,  $v_i(T|S_i) \leq \sum_{j \in T} p_j$ , since otherwise player  $j$  can strictly increase utility in the SPIB game by bidding very high on the items in  $T$ . This shows outward stability of  $(S_1, \dots, S_n)$  and  $(p_1, \dots, p_n)$ , showing that it is a CE.

Now suppose that there exists a CE  $(S_1, \dots, S_n)$  and  $(p_1, \dots, p_n)$  for  $(v_1, \dots, v_n)$ . We claim that the following bid vector  $\vec{b}$  is a pure Nash equilibrium with weak no-overbidding for SPIB with  $(v_1, \dots, v_n)$ .

$$b_i^j = \begin{cases} p_j & j \in S_i \\ 0 & j \notin S_i \end{cases}$$

Note that with this bid vector, each player  $i$  wins  $S_i$ , and pays zero. Clearly player  $i$  cannot increase her utility by changing any bid for an item in  $S_i$ , as she pays zero for these items. She also cannot increase utility by bidding higher on items in some subset  $T \subset \Omega \setminus S_i$ , as she will have to additionally pay  $\sum_{j \in T} p_j \geq v_i(T|S_i)$  for these items, where the inequality follows from outward stability. Thus  $\vec{b}$  is indeed a Nash equilibrium. By individual rationality,  $v_i(S_i) \geq \sum_{j \in S_i} p_j$ , implying that  $\vec{b}$  satisfies weak no-overbidding, and the claim follows.  $\square$

A main question studied by Bhawalkar and Roughgarden (2011) is the *price of anarchy* of the SPIB game, which is the ratio between the optimal social welfare and the lowest social welfare of any Nash equilibrium of the game. They show that the price of anarchy is 2 if  $v_1, \dots, v_n$  are subadditive. An immediate corollary of propositions 1 and 4 is that this holds *without any restriction on the valuation functions*.

**Corollary 2.** *For any  $(v_1, \dots, v_n)$ , the welfare of every pure Nash equilibrium with weak no-overbidding in the SPIB game is at least half of the optimal social welfare for  $(v_1, \dots, v_n)$ .*

The work of Bhawalkar and Roughgarden (2011) does not study the *existence* of a pure Nash equilibrium with weak no-overbidding in the SPIB game, and in this paper, by studying the existence of CE, we shed some light on this question, showing quite surprisingly that pure NE of this game are guaranteed to exist even when the valuations exhibit some (limited!) complementarities.

Let us briefly summarize the picture that we have portrayed up to now. Kelso and Crawford (1982) show that a Walrasian equilibrium exists for gross substitutes. Our goal is to study cases where gross substitutes is violated. For this purpose we have defined the notion of a conditional equilibrium, which is a relaxation of Walrasian equilibrium. This notion guarantees an approximate version of the First Welfare Theorem, and a conditional equilibrium always exists when valuations are marginally decreasing, although this is one of the cases where Walrasian equilibrium might not exist. Furthermore, the natural ascending-price process finds it, and the welfare-maximizing allocation is supported in a CE. There are also tight connections between conditional equilibria and Nash equilibria of a natural combinatorial auction with item bidding. We therefore wish to proceed with a more general analysis that aims to characterize the largest class of valuations that always admit a CE. We proceed gradually towards this goal, describing as the next step a strictly larger class of valuations for which all of the above properties continue to hold.

### 3 Fractionally Subadditive Valuations and the Flexible-Ascent Auction

When items are divisible, decreasing marginal valuations correspond to the convexity of the utility function, while concave utilities represent complementarities among items. When items are indivisible, as is our case, the situation is not so straight-forward. A valuation  $v$  exhibits explicit complementarities if there exist  $S, T \subseteq \Omega$  such that  $v(S \cup T) > v(S) + v(T)$ . Thus a valuation has no complementarities if it is subadditive, i.e.,  $v(S \cup T) \leq v(S) + v(T)$  for all  $S, T \subseteq \Omega$ . A valuation that is not marginally decreasing need not necessarily exhibit complementarities, as there are valuations that are subadditive but not marginally decreasing. In this section we show that all the appealing properties of a conditional equilibrium continue to hold for the class of “fractionally subadditive” valuations, that strictly contains the class of marginally decreasing valuations and is strictly contained in the class of subadditive valuations (Lehmann et al., 2006).

Formally, for any  $S \subseteq \Omega$ , a vector of non-negative weights  $\{\lambda_T\}_{T \subseteq S, T \neq \emptyset}$  is a “fractional cover” of  $S$  if for any  $x \in S$ ,  $\sum_{T \subseteq S \text{ s.t. } x \in T} \lambda_T = 1$ . A valuation  $v$  is “fractionally sub-additive” if for any  $S \subseteq \Omega$  and any fractional cover  $\{\lambda_T\}_{T \subseteq S, T \neq \emptyset}$  of  $S$ ,  $v(S) \leq \sum_{T \subseteq S, T \neq \emptyset} \lambda_T v(T)$ . The class of fractionally subadditive valuations is defined in Feige (2006), who also shows its equivalence to the class of XOS valuations defined in Nisan (2000).<sup>6</sup> As mentioned earlier the class of fractionally

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<sup>6</sup>A valuation is XOS if it is the maximum of several additive valuations.

subadditive valuations strictly expands the class of marginally decreasing valuations, for example the valuation  $\tilde{u}$  in Figure 1 is fractionally subadditive but it is not marginally decreasing.

There is an interesting connection between fractionally subadditive valuations and coalitional *cost* games. Viewing the pair  $(\Omega, v)$  as a coalitional cost game, the weights  $\{\lambda_T\}$  are a “fractional cover” of  $S$  if and only if they are “anti-balanced” in the subgame  $(S, v)$ , in the sense of the Bondareva-Shapley theorem. Thus,  $v$  is fractionally subadditive if and only if  $(\Omega, v)$  is totally anti-balanced. Recall that the “anti-core” of a coalitional cost game  $(\Omega, v)$  is a vector of prices  $\{p_x\}_{x \in \Omega}$  such that (1)  $\sum_{x \in \Omega} p_x = v(\Omega)$ , and (2) for any  $S \subseteq \Omega$ ,  $\sum_{x \in S} p_x \leq v(S)$ . Note that these are exactly supporting prices for  $\Omega$ . By the Bondareva-Shapley theorem,  $v$  is fractionally subadditive if and only if every subgame  $(S, v)$  has a non-empty anti-core, where the latter statement is equivalent to saying that there exist supporting prices for  $v(S)$  for every  $S \subseteq \Omega$ .<sup>7</sup> Thus, proposition 3 immediately implies:

**Proposition 5.** *A conditional equilibrium always exists for any tuple of fractionally subadditive valuations. In particular, the welfare-maximizing allocation is always supported by a conditional equilibrium.*

Moreover, one can find a conditional equilibrium using an ascending-price process. To show this, we use the following algorithm, which was suggested by Christodoulou et al. (2008) in a different context.<sup>8</sup>

**Definition 4** (The Flexible-Ascent Auction). Set initial item prices  $p_x$  ( $x \in \Omega$ ) to zero, and initial allocation  $S_1, \dots, S_n$  to be empty ( $S_i = \emptyset$ ). Then repeat:

1. Choose a player  $i$  with a non-empty conditional demand, i.e., such that there exists  $T \in \arg\max_{T \subseteq \Omega \setminus S_i} (v_i(T|S_i) - p(T))$ , and  $v_i(T|S_i) > p(T)$ .
2. Find prices  $\{p'_x\}_{x \in S_i \cup T}$  such that  $p'(S_i \cup T) > p(S_i \cup T)$ . Set  $p_x \leftarrow p'_x$  for every  $x \in S_i \cup T$ . Note that item prices may increase or decrease, we only require that the *sum* of item prices increases. Also note that it is not fully defined how to set  $p'$ , we give several options in the sequel.
3. Move items in  $T$  to  $i$ 's current allocation, i.e.,  $S_i \leftarrow S_i \cup T$ ,  $S_j \leftarrow S_j \setminus T$  for any  $j \neq i$ . (We say that player  $i$  “takes” items  $S_i \cup T$  in this step.)

... until all players have empty conditional demands.

We have intentionally left unspecified the price update rule, as we use in the sequel several such rules. However to argue that the auction terminates in a finite number of steps we need to assume

<sup>7</sup>The characterization in terms of supporting prices was suggested by Dobzinski et al. (2005), and the comparison with cooperative games was later discussed in Feige (2006).

<sup>8</sup>In Christodoulou et al. (2008) this algorithm is used to find a Nash equilibrium of a certain normal-form auction game, a purpose seemingly unrelated to ours.

something about the price update rule, since in general this may not be correct.<sup>9</sup> In particular, all the price update rules we define in the sequel determine prices in any step of the auction solely as a function of the player who takes new items in that step and the set of items that player receives. More precisely, if player  $i$  takes set  $T$  in a step of the auction, finishing the step with a tentative allocation of  $S_i \cup T$ , then the price  $p'_x$  for items  $x \in S_i \cup T$  will depend only on the index  $i$ , the valuation  $v_i$ , and the set  $S_i \cup T$ . Consequently, for a fixed profile of valuations  $(v_1, \dots, v_n)$ , every item  $x$  has only finitely many possible prices that may be set during the Flexible-Ascent auction. This implies that there are finitely many price vectors that can occur during the auction. Since the sum of item prices strictly increases during the auction, each price vector can occur at most once and the auction terminates after finitely many steps.

Outward stability of the end outcome of the auction is guaranteed by definition since at the end all players have empty conditional demands. Individual rationality again depends on the specific price update rule that we choose. For fractionally subadditive valuations, setting  $p'$  in step 2 to be supporting prices for  $v_i(S_i \cup T)$  ensures the individual rationality of the tentative allocation for every player in every step since if at some later step  $i$ 's tentative set of items was decreased to  $X \subseteq S_i \cup T$ , we have that the new prices at the later step  $p(X)$  are equal to  $p'(X)$  since no other player took the items in  $X$ , and by definition of supporting prices  $v_i(X) \geq p'(X)$ . This update rule (say from  $p$  to  $p'$ ) also satisfies the requirement to increase the sum of prices since  $p'(S_i \cup T) = v_i(S_i \cup T) > p(S_i \cup T)$ . Thus for fractionally subadditive valuations, setting  $p'$  in step 2 to be supporting prices ensures that the end outcome is a CE. All of this implies the following proposition.

**Proposition 6.** *The Flexible-Ascent auction terminates in a conditional equilibrium, for any tuple of fractionally subadditive valuations.*

These properties motivate some general characterization questions that we discuss in the next section.

## 4 Towards a Complete Characterization of Existence Classes

Given the encouraging results of the previous sections, one may wonder how large is the class of valuations that admits some or all of the positive properties described above. In particular, we are interested in the following questions:

- Can a conditional equilibrium exist when valuations exhibit a mixture of substitutabilities and complementarities? If so, what is the largest class of valuations that always admit a conditional equilibrium?

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<sup>9</sup>For example, with one item, the auction will not terminate if players update the price from  $p$  to  $(p + v)/2$ , and there exist two players with the same highest valuation. A simple solution is to assume that all valuations and prices are integers, but with our price update rules the auction will terminate after a finite number of steps even with real valuations.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\tilde{u}$	5	3	3	6	6	6	9
$\tilde{v}$	3	3	3	6	6	4	8

Figure 1: An example with four items  $\Omega = \{a, b, c, d\}$  and two players with valuations  $\tilde{u}, \tilde{v}$ . The table describes all values for nonempty subsets of  $\{a, b, c\}$ . We additionally have (1)  $\tilde{u}(d) = 1.1$  and  $\forall S \subseteq \Omega \setminus \{d\}, S \neq \emptyset, \tilde{u}(d|S) = 0$ , (2)  $\forall S \subseteq \Omega, \tilde{v}(d|S) = 0$ . Allocating all items to  $\tilde{u}$  with prices  $p_a = p_b = p_c = 3$  and  $p_d = 0$  is a CE.

- Does the existence of a conditional equilibrium imply that it can be reached via an ascending-price process?
- Does the existence of a conditional equilibrium imply that the welfare-maximizing allocation is supported by a conditional equilibrium? In other words, does the second welfare theorem hold whenever a conditional equilibrium exists?

Figure 1 shows a constructive example that demonstrates that the answer to the first question is positive, and in this section we will mainly engage in a formal mathematical treatment of this question. In the example, the valuation  $\tilde{v}$  is not subadditive, since  $\tilde{v}(abc) = 8 > 3 + 4 = \tilde{v}(a) + \tilde{v}(bc)$ . Thus the example demonstrates the existence of a CE in a setting with complementarities. The valuation  $\tilde{u}$  is fractionally subadditive but it does not have decreasing marginal values since  $\tilde{u}(b|cd) = 3 > 1.9 = \tilde{u}(b|d)$ .<sup>10</sup> We show below that it is possible to have a CE even if all valuations are not subadditive.

Perhaps even more interestingly, the example gives a negative answer to our third question, demonstrating that the welfare-maximizing allocation need not be a CE, *even if a CE exists*. In particular, in the example, the unique welfare-maximizing allocation is  $\{d\}$  to player  $\tilde{u}$  and  $\{a, b, c\}$  to player  $\tilde{v}$ . To see that this allocation cannot admit a CE, note that prices  $p$  that would admit a CE for this allocation must satisfy  $p_a \geq 3.9$  (since  $\tilde{u}(a|d) = 3.9$ ) and  $p_b + p_c \geq 4.9$  (since  $\tilde{u}(bc|d) = 4.9$ ). However we also need  $p_a + p_b + p_c \leq \tilde{v}(abc) = 8$ , a contradiction.

We are left with the first two questions. Our main focus in this section is the existence question, that seems more fundamental, though for the existence results we mainly use the Flexible-Ascent auction, hence we provide some insights regarding the second question as well.

We say that a class of valuations  $X$  admits a CE if any tuple of valuations  $(v_1, \dots, v_n) \in X^n$  admits a CE. Following Gul and Stacchetti (1999), we are interested in classes that contain unit-demand valuations. A valuation  $v_i$  is unit-demand if  $v_i(S_i) = \max_{x \in S_i} v_i(x)$  for any  $S_i \subseteq \Omega$  — the interpretation is that a unit-demand player is interested only in a single item. We remark that our results hold if we replace unit-demand valuations by other possible fundamental valuation classes,

<sup>10</sup>One way to verify that  $\tilde{u}$  is fractionally subadditive is via its XOS formula  $\tilde{u} = (a = 3 \text{ OR } b = 3 \text{ OR } c = 3) \text{ XOR } (a = 5) \text{ XOR } (d = 1.1)$ .

for example, additive valuations where  $v_i(S_i) = \sum_{x \in S_i} v_i(x)$  for any  $S_i \subseteq \Omega$ . Define the MAXCE requirements to be the following criteria for a valuation class that we generically refer to as  $V^{CE}$ :

- All unit-demand valuations belong to  $V^{CE}$ .
- For all  $n \geq 1$ , any tuple  $v = (v_1, \dots, v_n) \in (V^{CE})^n$  admits a CE.
- For any  $u \notin V^{CE}$  there exist  $v_1, \dots, v_k \in V^{CE} \cup \{u\}$  (for some integer  $k$ ) such that  $(v_1, \dots, v_k)$  does not admit a CE.

Gul and Stacchetti (1999) show that the class of gross-substitutes (GS) is the unique class that satisfies these conditions when considering a Walrasian equilibrium instead of a conditional equilibrium. Since a Walrasian equilibrium is also a conditional equilibrium, Zorn’s Lemma implies that there exists a class  $V^{CE}$  that contains all gross-substitutes and satisfies the MAXCE requirements. In fact our results from the previous sections imply that there exists a class  $V^{CE}$  that contains all fractionally subadditive valuations and satisfies the MAXCE requirements.

**Main Question:** *Describe explicitly a valuation class satisfying the MAXCE requirements. Is there a unique such class?*

As a side remark, we wish to mention an additional motivation for this question, originating from its possible implications for the search for computationally efficient algorithms that approximate the optimal social welfare in combinatorial auctions. Computing the optimal social welfare is NP-hard even if decreasing marginal values are assumed, and many computationally efficient approximation algorithms were constructed for this problem.<sup>11</sup> However constant-factor approximations are known only for subadditive valuations (Feige, 2006). Since we show that  $V^{CE}$  can contain non-subadditive valuations as well, and since a CE guarantees a 2-approximation of the optimal social welfare, one may envision a computationally efficient 2-approximation for valuations in  $V^{CE}$  by efficiently computing a CE. Thus, if  $V^{CE}$  will turn out to contain “many” non-subadditive valuations, our main question will have implications on the important search for computationally-efficient algorithms that approximate the optimal social welfare in combinatorial auctions.

#### 4.1 An upper bound on any $V^{CE}$

We prove that the following class of valuations upper bounds any  $V^{CE}$ .

**Definition 5** ( $\overline{V}^{CE}$ ). A valuation function  $v_i$  belongs to  $\overline{V}^{CE}$  if and only if for any  $S \subseteq \Omega$  with  $|S| \geq 2$ ,  $v_i(S) \leq \frac{1}{|S|-1} \sum_{x \in S} v_i(S \setminus x)$ .

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<sup>11</sup>Given a tuple of valuations  $v$ , a  $c$ -approximation algorithm outputs an allocation whose social welfare is at least  $1/c$  times the optimal social welfare for  $v$ .

When  $|\Omega| = 2$ ,  $\bar{V}^{CE}$  is simply the class of all subadditive valuations, as the (single) requirement in this case is  $v_i(\Omega) \leq \sum_{x \in \Omega} v_i(x)$ . For a general  $\Omega$ , first observe that  $\bar{V}^{CE}$  contains all fractionally-subadditive valuations since one fractional cover for any given set  $S \subseteq \Omega$  is the collection of subsets  $\{S \setminus x\}_{x \in S}$  with equal weights  $\frac{1}{|S|-1}$ . Thus if  $v$  is fractionally-subadditive then  $v_i(S) \leq \frac{1}{|S|-1} \sum_{x \in S} v_i(S \setminus x)$ , which implies that  $v \in \bar{V}^{CE}$ . However, when  $|\Omega| > 2$ , the location in the valuation hierarchy is not so standard:  $\bar{V}^{CE}$  contains valuations that are not subadditive, for example one can verify that the valuation  $\tilde{v}$  in Figure 1 is in  $\bar{V}^{CE}$ . On the other hand, it can be shown that  $\bar{V}^{CE}$  does not contain all subadditive valuations. More specifically, it can be shown that  $\bar{V}^{CE}$  fails to contain some subadditive valuations that are not fractionally-subadditive. (In fact, when  $|\Omega| = 3$ , all such valuations lie outside  $\bar{V}^{CE}$ .)

We show that any  $V^{CE} \subseteq \bar{V}^{CE}$ . We start by observing two useful properties of valuations in  $\bar{V}^{CE}$ .

**Lemma 1.**

1.  $v \in \bar{V}^{CE}$  if and only if  $\forall S \subseteq \Omega$ ,  $\sum_{x \in S} v(x|S \setminus x) \leq v(S)$ .
2. For any  $v \in \bar{V}^{CE}$  and  $S \subseteq \Omega$ ,  $v(S) \leq \sum_{x \in S} v(x)$ .

*Proof.* The first part of the lemma is an immediate consequence of the following manipulation.

$$\sum_{x \in S} v(x|S \setminus x) = \sum_{x \in S} v(S) - v(S \setminus x) = |S| \cdot v(S) - \sum_{x \in S} v(S \setminus x).$$

The second part is proven by induction on  $|S|$ , the base case being trivial. For  $|S| > 1$ ,

$$v(S) \leq \frac{\sum_{x \in S} v(S \setminus x)}{|S| - 1} \leq \frac{\sum_{x \in S} \sum_{y \in S \setminus x} v(y)}{|S| - 1} = \sum_{y \in S} v(y),$$

where the second inequality follows from the induction hypothesis. □

**Theorem 1.** *If  $V^{CE}$  satisfies the MAXCE requirements then  $V^{CE} \subseteq \bar{V}^{CE}$ . In particular, for any  $u \notin \bar{V}^{CE}$ , there exist unit-demand valuations  $v_1, \dots, v_k$  such that  $(u, v_1, \dots, v_k)$  does not admit a CE.*

*Proof.* Since  $u \notin \bar{V}^{CE}$ , Lemma 1 implies that there exists  $S \subseteq \Omega$  such that  $\sum_{x \in S} u(x|S \setminus x) > u(S)$ . We construct the following tuple of unit-demand valuations. For every item  $x \in \Omega \setminus S$  we have two unit-demand valuations  $v_x^{(1)} = v_x^{(2)}$  such that  $v_x^{(i)}(x) = u(\Omega) + 1$  and  $v_x^{(i)}(y) = 0$  for any item  $y \neq x$ . Additionally define a unit-demand valuation  $v_S$  as follows. Choose a small enough  $\epsilon > 0$  such that (i)  $\sum_{x \in S} (u(x|S \setminus x) - \epsilon) > u(S)$ , and (ii)  $\forall x \in S$  such that  $u(x|S \setminus x) > 0$ ,  $\epsilon < u(x|S \setminus x)$ . Then define

$$v_S(x) = \begin{cases} \max(0, u(x|S \setminus x) - \epsilon) & x \in S \\ 0 & x \notin S \end{cases}$$

We show that there does not exist a CE for the valuation  $u$  coupled with these unit-demand valuations. Note that in every possible CE for these valuations, every item  $x \in \Omega \setminus S$  must be allocated to either player  $v_x^{(1)}$  or  $v_x^{(2)}$  and its price must be  $v_x^{(1)}(x)$ . As a result, note that if in some CE outcome a player  $v_x^{(i)}$  is being allocated some item  $y \in S$ ,  $y$ 's price must be zero. Suppose by contradiction that there exists a CE with prices  $p$  and in which player  $u$  is allocated a set of items  $T_u$ , player  $v_S$  is allocated a set of items  $T_v$ , and  $T_u \cup T_v \subseteq S$ .

If  $T_v = \emptyset$  or  $v_S(T_v) = 0$  (in which case  $p(T_v)$  is 0), then we have  $p(S) = \sum_{x \in T_u} p_x \leq u(T_u) \leq u(S) < \sum_{x \in S} (u(x|S \setminus x) - \epsilon)$ . Thus, there exists an item  $x \in S \setminus T_v$  with  $p_x < u(x|S \setminus x) - \epsilon \leq v_S(x)$ . Since in this case player  $v_S$  will desire such an item  $x$ , this cannot be a CE.

Otherwise,  $v_S(T_v) > 0$ . Let  $x^* = \operatorname{argmax}_{x \in T_v} v_S(x)$ , then  $v_S(x^*) = u(x^*|S \setminus x^*) - \epsilon$ . Since  $p(S \setminus T_u) = p(T_v) \leq v_S(x^*)$  we have,

$$\begin{aligned} u(S \setminus T_u|T_u) - p(S \setminus T_u) &> u(S \setminus T_u|T_u) - u(x^*|S \setminus x^*) \\ &= (u(S) - u(T_u)) - (u(S) - u(S \setminus x^*)) \\ &= u(S \setminus x^*) - u(T_u) \geq 0, \end{aligned}$$

where the last inequality follows since  $T_u \subseteq S \setminus x^*$ . This inequality contradicts the outward-stability property of a CE, and we conclude that there does not exist a CE for the valuation  $u$  coupled with the unit-demand valuations defined above.  $\square$

Does there always exist a CE for valuations in  $\overline{V}^{CE}$ ? We do not have a counterexample, nor a general proof, and this is left open for future research. Some very partial indication that the answer may be positive can be drawn from the case of three items, for which we prove in the next section a positive answer. Another indication is the special case where one bidder has valuation in  $\overline{V}^{CE}$  and all other bidders have submodular valuations. The following theorem gives a positive answer to this case as well.

**Theorem 2.** *For any valuation  $v_1 \in \overline{V}^{CE}$ , and submodular valuations  $v_2, \dots, v_n$ , there exists a CE. In fact, for any social welfare maximizing allocation, there exists a price vector under which the allocation constitutes a CE.*

*Proof.* Let  $(S_1, S_2, \dots, S_n)$  be any allocation that maximizes the social welfare for these valuations. We show that there exists price vector  $p$  such that  $(S_1, \dots, S_n)$  constitutes a CE under these prices. Since  $v_2, \dots, v_n$  are submodular, for each  $S_i$ ,  $2 \leq i \leq n$ , there exists a supporting price  $p^i$  for  $v_i(S_i)$ . We define  $p$  to be

$$p(x) = \begin{cases} v_1(x | S_1 \setminus x), & \text{for } x \in S_1; \\ p^i(x), & \text{for } x \in S_i, 2 \leq i \leq n. \end{cases}$$

Individual rationality follows from Lemma 1 (in the case of  $v_1$ ) and from the definition of supporting prices (in the cases of  $v_2, \dots, v_n$ ). It remains only to show that outward stability holds.

For player 1, fix any non-empty  $T \subseteq \Omega - S_1$ . Let  $R_i = T \cap S_i$ . Since  $(S_1, \dots, S_n)$  maximize the social welfare,

$$v_1(S_1) + \sum_{2 \leq i \leq n} v_i(S_i) \geq v_1(T \cup S_1) + \sum_{2 \leq i \leq n} v_i(S_i \setminus R_i),$$

or

$$v_1(T | S_1) \leq \sum_{2 \leq i \leq n} v_i(R_i | S_i \setminus R_i) \leq \sum_{2 \leq i \leq n} v_i(R_i) = \sum_{2 \leq i \leq n} p^i(R_i) = p(T).$$

For any other player  $i$ ,  $2 \leq i \leq n$ , since  $v_i$  is submodular, if his conditional demand is nonempty, then there exists an item  $x \in \Omega \setminus S_i$  such that  $v_i(x | S_i)$  exceeds the price  $p(x)$ . However,  $x$  cannot be in  $S_1$ , since that would mean  $v_i(x \cup S_i) - v_i(S_i) > p(x) = v_1(S_1) - v_1(S_1 \setminus x)$ , which says that if we just move  $x$  from  $S_1$  to  $S_i$ , the social welfare would strictly increase, contradicting our earlier hypothesis that  $(S_1, \dots, S_n)$  maximize the social welfare. By a similar argument, one can show that  $x$  cannot be in some other  $S_j$ , either. Therefore, outward stability is satisfied for all players.  $\square$

It is worth noting that when one attempts to assume one valuation in  $\overline{V}^{CE}$  and  $n-1$  fractionally subadditive valuations, a similar theorem no longer holds, as shown by the two valuations in Figure 1. In fact, we do not know if in this case a CE always exists.

## 4.2 A lower bound on at least one $V^{CE}$

We already know that there exists a  $V^{CE}$  that contains all fractionally subadditive valuations. This section slightly extends this lower bound, as follows. A “restriction” of some valuation  $v$  to a subset of items  $S \subseteq \Omega$ , denoted by  $v|_S$ , is a valuation over the items in  $S$  such that  $v|_S(T) = v(T)$  for any  $T \subseteq S$ . Note that a valuation is fractionally subadditive if and only if for any  $S \subseteq \Omega$ ,  $v|_S$  is fractionally subadditive.

**Definition 6** ( $\underline{V}^{CE}$ ). A valuation  $v$  belongs to  $\underline{V}^{CE}$  if and only if  $v \in \overline{V}^{CE}$ , and for any  $S \neq \Omega$ ,  $v|_S$  is fractionally subadditive.

Note that  $\underline{V}^{CE} \subseteq \overline{V}^{CE}$ , and that  $\underline{V}^{CE}$  contains all fractionally subadditive valuations since  $\overline{V}^{CE}$  contains all these valuations.  $\underline{V}^{CE}$  also contains valuations that are not fractionally subadditive, for example, the following valuation  $v$  over three items  $\Omega = \{a, b, c\}$ :  $v(a) = v(b) = 4, v(c) = 2, v(ab) = v(ac) = v(bc) = 6, v(abc) = 9$ . It is a simple exercise to verify that this valuation is in  $\underline{V}^{CE}$ , but it is not subadditive since  $v(abc) > v(ab) + v(c)$ , and hence not fractionally subadditive. We show that every tuple of valuations that belong to  $\underline{V}^{CE}$  admits a CE:

**Theorem 3.** *There exists a valuation class  $V^{CE} \supseteq \underline{V}^{CE}$  satisfying the MAXCE requirements.*

*Proof.* Given a tuple of valuations  $v_1, \dots, v_n \in \underline{V}^{CE}$ , we show there exists a CE for  $(v_1, \dots, v_n)$ ; as before, the theorem then follows by applying Zorn’s Lemma. Fix a player  $i \in \operatorname{argmax}_{i=1, \dots, n} \{v_i(\Omega)\}$ . Suppose first that for every player  $j \neq i$  and for every item  $x \in \Omega$ ,  $v_j(x) \leq v_i(x | \Omega \setminus \{x\})$ . In this case allocating all items to player  $i$  and setting  $p_x = v_i(x | \Omega \setminus \{x\})$  for every  $x \in \Omega$  is a CE, since:

- Individual rationality follows since  $v_i(\Omega) \geq \sum_{x \in \Omega} v_i(x|\Omega \setminus \{x\}) = p(\Omega)$ , where the inequality is by Lemma 1.
- Outward stability follows since, for every player  $j \neq i$  and for every subset  $S \subseteq \Omega$ ,

$$v_j(S) \leq \sum_{x \in S} v_j(x) \leq \sum_{x \in S} v_i(x|\Omega \setminus \{x\}) = p(S),$$

where the first inequality is again by Lemma 1.

Hence in this case a CE exists. Now assume that there exists a player  $j$  and an item  $x$  such that  $v_j(x) > v_i(x|\Omega \setminus \{x\})$ . In this case, to prove the existence of a CE, we run the Flexible-Ascent auction with a different initial condition:

- Player  $i$ 's initial assignment is  $\Omega \setminus \{x\}$  and the prices of these items are supporting prices for  $v_i(\Omega \setminus \{x\})$  (recall that such prices exist as  $v_i|_{\Omega \setminus \{x\}}$  is fractionally subadditive).
- Player  $j$ 's initial assignment is  $\{x\}$ , and the price of item  $x$  is  $v_j(x)$ .

Note that sum of initial prices is strictly larger than  $v_i(\Omega) \geq v_{i'}(\Omega)$  for any player  $i'$ . Since the sum of prices cannot decrease during the auction, we conclude that no player takes  $\Omega$  during the auction. Therefore we are able to set supporting prices to the set of items taken in every step, which implies that the Flexible-Ascent auction terminates and the end outcome is a CE.  $\square$

It is rather straightforward to verify that  $\underline{V}^{CE} = \overline{V}^{CE}$  when  $|\Omega| = 2$  and when  $|\Omega| = 3$ . Thus for these cases we have a complete characterization:

**Corollary 3.** *When  $|\Omega| \leq 3$ , the unique valuation class  $V^{CE}$  that satisfies the MAXCE requirements is  $V^{CE} = \underline{V}^{CE} = \overline{V}^{CE}$ .*

With four items or more,  $\underline{V}^{CE} \neq \overline{V}^{CE}$ . For example, the valuation  $\tilde{v}$  in Figure 1 belongs to  $\overline{V}^{CE}$  but not to  $\underline{V}^{CE}$  since  $\tilde{v}|_{\{a,b,c\}}$  is not subadditive. Hence our bounds are not tight in general. In particular, as we next discuss, most probably our lower bound is too weak.

### 4.3 Some evidence that $V^{CE}$ may be different than $\underline{V}^{CE}$

We show that, if we restrict attention to two players and four items, there exists a  $V^{CE}$  that strictly contains  $\underline{V}^{CE}$ . More specifically, we show that there exist valuations  $v_2 \in \overline{V}^{CE} \setminus \underline{V}^{CE}$  such that, for *any* valuation  $v_1 \in \overline{V}^{CE}$ , there exists a CE for the tuple of valuations  $(v_1, v_2)$ . Such a valuation  $v_2$ , is, for example, the valuation  $\tilde{v}$  defined in Figure 1. This implies, at least when  $n = 2$  and  $|\Omega| = 4$ , that  $\underline{V}^{CE}$  is not maximal, since such a valuation  $v_2$  can be added to it. We conclude that there is some fundamental structure of  $V^{CE}$  that the previous analysis does not capture. We leave this as an open problem for future research.

**Theorem 4.** Suppose  $\Omega = \{a, b, c, d\}$ , and fix any  $v_2 \in \overline{V}^{CE}$  such that, for any  $S \subseteq \{a, b, c\}$ ,  $v_2(d|S) = 0$ . Then, for any valuation  $v_1 \in \overline{V}^{CE}$ , there exists a CE for  $(v_1, v_2)$ .<sup>12</sup>

*Proof.* Regarding notation, player 1 has valuation  $v_1$  and player 2 has valuation  $v_2$ . We need to prove that there exists a CE for  $(v_1, v_2)$ . We consider three cases.

**Case 1:**  $v_2(\Omega) \geq \max\{v_1(S_1) + v_2(S_2) \mid S_2 \neq \{a, b, c\}, S_1 \cap S_2 = \emptyset\}$ . In this case we show that allocating  $\{d\}$  to player 1 and  $\{a, b, c\}$  to player 2, with prices  $p_x = v_1(x)$  for every item  $x \in \Omega$ , is a CE.

- **Outward Stability.** By definition  $v_2(d|abc) = 0 \leq p_d$ , implying outward stability for player 2. For every  $S \subseteq \{a, b, c\}$  we have by Lemma 1 that  $v_1(S \cup \{d\}) \leq \sum_{x \in S} v_1(x) + v_1(d) = p(S) + v_1(d)$ . This implies that  $v_1(S|d) = v_1(S \cup \{d\}) - v_1(d) \leq p(S)$ , implying outward stability for player 1.
- **Individual Rationality.** IR is immediate for player 1; we show it for player 2. For every  $S \subseteq \Omega$  such that  $d \in S$  and  $|S| = 2$ , we have  $v_2(abc) = v_2(abcd) \geq v_1(S) + v_2(\Omega \setminus S)$ . Thus

$$\begin{aligned} 3v_2(abc) &\geq v_1(ad) + v_1(bd) + v_1(cd) + v_2(ab) + v_2(ac) + v_2(bc) \\ &\geq v_1(ad) + v_1(bd) + v_1(cd) + 2v_2(abc), \end{aligned}$$

where the second inequality follows since one can verify that  $v_2 \in \overline{V}^{CE}$  and therefore by definition  $2v_2(abc) \leq v_2(ab) + v_2(ac) + v_2(bc)$ . This implies  $v_2(abc) \geq v_1(ad) + v_1(bd) + v_1(cd) \geq p_a + p_b + p_c$  as needed.

**Case 2:**  $v_1(\Omega) \geq \max\{v_1(S_1) + v_2(S_2) \mid S_2 \neq \{a, b, c\}, S_1 \cap S_2 = \emptyset\}$ . In this case we show that allocating all items to player 1, with item prices  $p_x = v_1(x|\Omega \setminus x)$  for every  $x \in \Omega$ , is a CE. Individual rationality for player 1 follows since  $v_1(\Omega) \geq \sum_{x \in \Omega} v_1(x|\Omega \setminus x)$  by Lemma 1, and for player 2 individual rationality is empty. Outward stability for player 1 is empty. It remains to show that  $v_2(S) \leq p(S)$  for every  $S \subseteq \Omega$ , which implies outward stability for player 2. By assumption, for any  $x \in \Omega$ ,  $v_1(\Omega) \geq v_2(x) + v_1(\Omega \setminus x)$ . Thus  $p_x = v_1(x|\Omega \setminus x) \geq v_2(x)$ , and as before  $v_2(S) \leq \sum_{x \in S} v_2(x) \leq p(S)$ .

**Case 3:** There exists an allocation  $(S_1, S_2)$  such that  $S_2 \neq \{a, b, c\}$  and  $v_1(S_1) + v_2(S_2) > \max_{i=1,2} v_i(\Omega)$ . Without loss of generality assume that  $d \in S_1$ . This implies that  $|S_2| \leq 2$ . In this case we show that the Flexible-Ascent auction terminates in a CE. We first prove a useful lemma:

**Lemma 2.** Fix any  $S \subseteq \Omega \setminus \{d\}$  such that  $1 \leq |S| \leq 2$  and any  $v_1 \in \overline{V}^{CE}$ . Then there exist prices  $\{p_x\}_{x \in S}$  such that  $p(S) = v_1(S \cup \{d\})$  and for every  $T \subseteq S$ ,  $p(T) \leq v_1(T \cup \{d\})$ .

<sup>12</sup>For example, the valuation  $\tilde{v}$  defined in Figure 1 satisfies the condition on  $v_2$ . It can be verified that this valuation is in  $\overline{V}^{CE} \setminus \underline{V}^{CE}$ .

*Proof.* If  $S = \{x\}$  we set  $p_x = v_1(\{x, d\})$  and the claim immediately holds. If  $S = \{x, y\}$  we set  $p_x = v_1(\{x, d\})$  and  $p_y = v_1(\{x, y, d\}) - v_1(\{x, d\})$ . To prove the claim it remains to argue that  $p_y \leq v_1(\{y, d\})$ . Lemma 1 implies

$$\begin{aligned} 2v_1(\{x, y, d\}) &\leq v_1(\{x, y\}) + v_1(\{x, d\}) + v_1(\{y, d\}) \\ &\leq v_1(\{x, y, d\}) + v_1(\{x, d\}) + v_1(\{y, d\}). \end{aligned}$$

This implies  $v_1(\{x, y, d\}) \leq v_1(\{x, d\}) + v_1(\{y, d\})$ , as needed. □

We now run the Flexible-Ascent auction, as follows:

- Initial allocation is the chosen  $(S_1, S_2)$  for this case. Initial price  $p_d = 0$ , initial prices of all other items in  $S_1$  are as determined by Lemma 2 above, and prices of all items in  $S_2$  are some supporting prices for  $v_2(S_2)$  (since  $|S_2| \leq 2$  we are guaranteed that supporting prices exist).
- Throughout the auction we maintain (1)  $p_d = 0$  and (2) item  $d$  remains allocated to player 1. We note that at initial prices  $p(abc) = v_1(S_1) + v_2(S_2) > \max(v_1(\Omega), v_2(\Omega))$ . Therefore player  $i = 1, 2$  never takes a subset  $S_i \cup T$  that contains  $\{a, b, c\}$  during the auction.
- Whenever player 1 takes  $S \cup T$  in the auction, prices  $p'$  are set according to Lemma 2 (since item  $d$  remains allocated to player 1 throughout, and  $p_d = 0$ ). This guarantees a strict increase of sum of prices in step 2 of the Flexible-Ascent auction since  $p'(S \cup T) = v_1(S \cup T)$ . By Lemma 2 this also guarantees individual rationality for player 1 throughout the auction. Whenever player 2 takes items, her new set of items is of size at most 2 since item  $d$  is never taken by her and since  $\{a, b, c\}$  cannot be taken. Thus prices in this case are set to be supporting prices for  $v_2$  of taken items, which again guarantees strict increase of sum of prices and individual rationality for player 2 throughout the auction.

Thus the Flexible-Ascent auction terminates in an outcome that satisfies individual rationality and outward stability, so it reaches a CE. This completes the proof of Theorem 4. □

## 5 Summary

Existence of a Walrasian equilibrium is limited to small valuation classes, relative to the entire valuation hierarchy. This motivates us to define a relaxed notion of a conditional equilibrium (CE), requiring that a player's value for her allocated bundle is larger than the bundle's price ("individual rationality"), and that the marginal value of any additional items is smaller than their total price ("outward stability"). This implies conditional stability. It also guarantees an approximate version of the first welfare theorem: in any conditional equilibrium, at least half of the optimal social

welfare is obtained. For marginally decreasing valuations, a conditional equilibrium always exists, while a Walrasian equilibrium need not. Furthermore, a conditional equilibrium can be obtained as a result of a natural ascending-price process, and the welfare-maximizing allocation is supported in a CE. We in fact show that all of these are true even for the larger class of fractionally subadditive valuations.

These results motivate us to study the characterization question of the existence of a CE. We ask what is a maximal class of valuations that admit a CE. We show lower and upper bounds on such classes. The upper bound is the class of all valuations  $v$  that satisfy, for any  $S \subseteq \Omega$  with  $|S| \geq 2$ , that  $v(S) \leq \frac{1}{|S|-1} \sum_{x \in S} v(S \setminus x)$ . This includes all fractionally subadditive valuations, as well as a rich structure of valuations with complementarities. The lower bound includes all valuations  $v$  that satisfy the previous condition and are fractionally subadditive when restricted to any strict subset of  $\Omega$ . This lower bound is probably too weak, and we leave the complete characterization as an open question for future research. However, even our partial characterization demonstrates that a conditional equilibrium exists for valuation classes that are significantly richer than gross substitutes, which we believe makes our question interesting, both conceptually and technically.

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