Learning Goals

- State the implementation of the Quicksort algorithm
- Define Las Vegas and Monte Carlo algorithms
- Basic analysis of the running time of randomized algorithms
- Develop intuitive understanding of the balls and bins asymptotics

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- Recall lower bound: no deterministic algorithm can make $o(n \log n)$ comparisons in the worst case.
- One of the best known sorting algorithm — Quicksort(S): If $|S| \leq 3$, return sorted S. Otherwise, pick an element a_i uniformly at random from S, form two sets: $S^+ := \{a_j : a_j > a_i\}$ and $S^- := \{a_j : a_j < a_i\}$. Return Quicksort($S^-$), a_j, Quicksort(S^+).

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- Later in the semester we will also encounter algorithms that give *approximations*, and we reason about the quality of the approximations in a probabilistic manner.

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- Observation: In each recursion, forming S^+ and S^- altogether takes $O(n)$ time.
- Intuition: if a_j always roughly cuts S in the middle, then the running time is roughly $T(n) \approx 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$.

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- The running time for each *level* in total is $O(n)$, so it suffices to show that, with high probability, the height of the tree is $O(\log n)$.

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- Let X be $\sum_{i=1}^{12 \log n} X_i$. By Chernoff bound, we have

$$\mathbf{Pr}\left[X < \frac{\log n}{\log \frac{4}{3}}\right] \leq \mathbf{Pr}\left[X < \mathbf{E}\left[X\right] - 5 \log n\right] \leq \exp\left(-2 \cdot \frac{25 \log^2 n}{12 \log n}\right) = n^{-25/6}.$$

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- Therefore, with high probability, the height of the tree is bounded by $12 \log n$.
- Obviously the constants in the analysis were not fine-tuned.

The Negative Binomial Distribution

- In the proof above, we wanted to bound the probability that we take more than $12 \log n$ steps to see $\log_{4/3} n$ good ones; instead, we bounded the probability that, within $12 \log n$ steps, there are fewer than $\log_{4/3} n$ good ones.

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- Answer: Yes. A random variable counting the number of i.i.d. trials before seeing k successful ones is said to follow the *negative binomial distribution*.
- The probability that such a random variable is larger than n is equal to the probability that, within n i.i.d. trials we have not seen k successful ones.
    - The statement may seem obvious, but a formal argument needs either "coupling" or some careful calculations.

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- Any bin receives in expectation $\frac{n}{m}$ balls. If $m = n$, this is 1.
- How about the bin that received the most balls? How many balls should we expect to see there?

Balls and Bins when $m = n$

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- For $t > 0$, we use Chernoff bound

$$\mathbf{Pr}\left[X > (1+t)\,\mathbf{E}\left[X\right]\right] \leq \left(\frac{e^t}{(1+t)^{1+t}}\right)^{\mathbf{E}[X]} \leq \left(\frac{e}{1+t}\right)^{1+t}.$$

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- We would like to find t so that this probability is smaller than n^{-2}. Essentially we are asking what solves $x^x = n$.

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- Note that $x < \log n$.
- We have $2 \log x \geq \log x + \log \log x = \log \log n \geq \log x$, so

$$\frac{1}{2} x \leq \frac{\log n}{\log \log n} \leq x \Rightarrow x = \Theta\left(\frac{\log n}{\log \log n}\right).$$

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- Let the solution to $x^x = n$ be $\gamma(n)$, and let $1 + t = e\gamma(n)$, we have

$$\left(\frac{e}{1+t}\right)^{1+t} = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} = n^{-e} < n^{-2}.$$

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- By union bound, with probability at least $1 - \frac{1}{n}$, no bin receives more than $e\gamma(n) = \Theta(\frac{\log n}{\log \log n})$ balls.

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Theorem

For $n = \Omega(m \log m)$, with high probability, the number of balls every bin receives is between half and twice the average.