Learning Goals

- Define a binary search tree
- Implement Insertion and Deletion on a binary search tree
- Implement tree rotations
- Understand the meaning and consequence of balancedness of a BST

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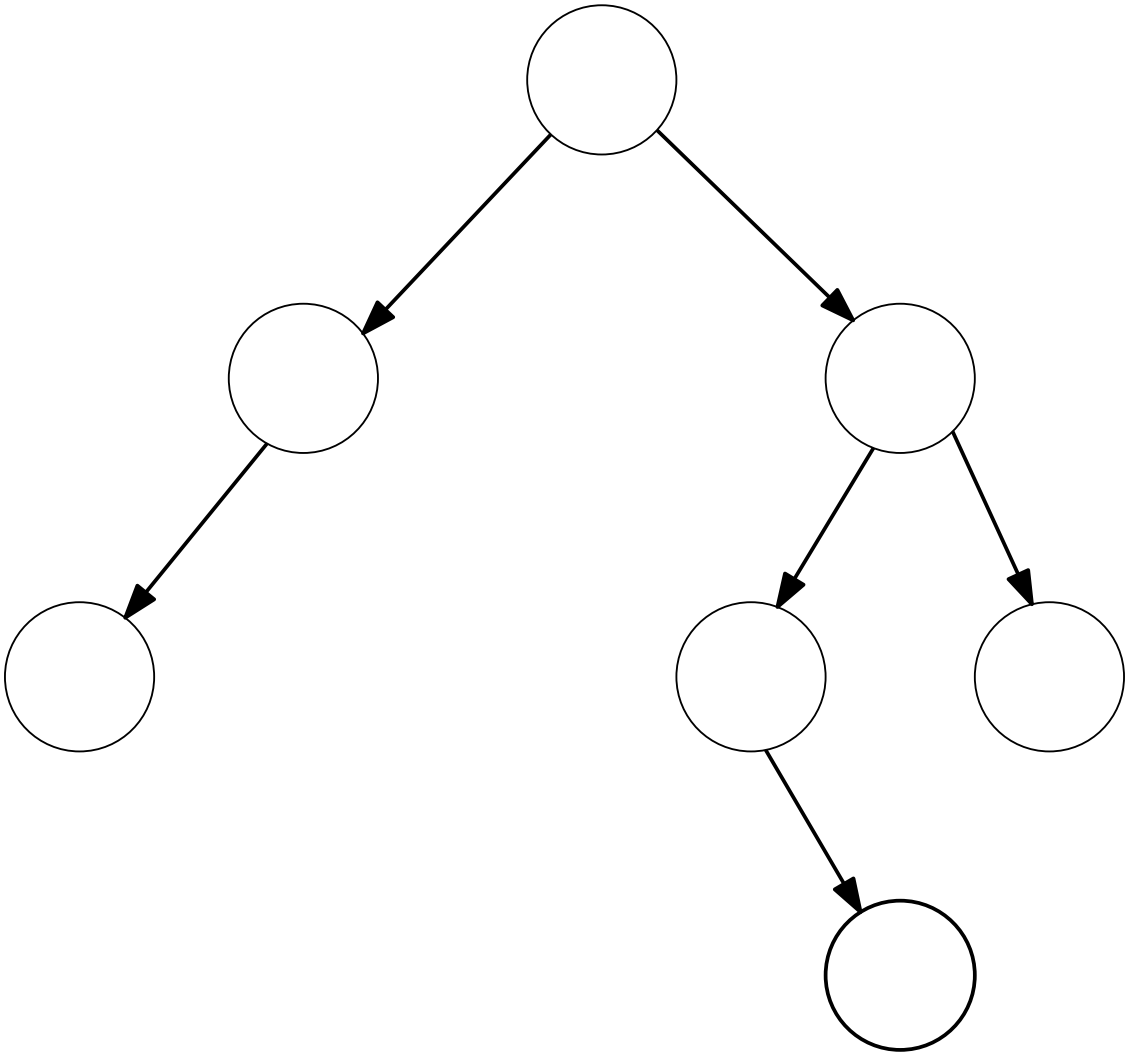
- A tree is *binary* if every node of it has at most 2 children.
  - For a binary tree, a node's two children are referred to as its LEFT child and its RIGHT child.

Illustration: A Binary Tree



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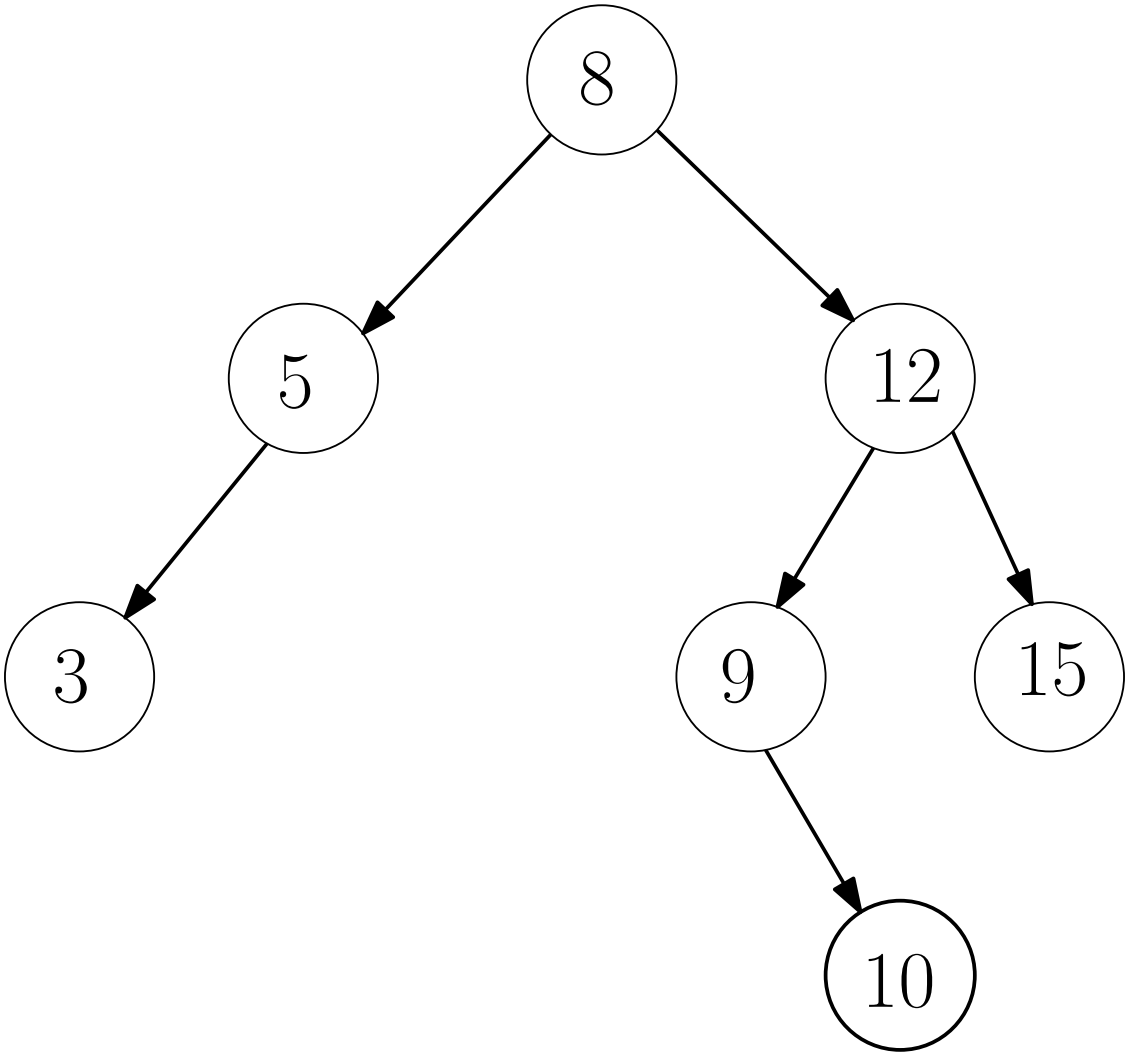
- A binary tree is a *binary search tree* if for every node in it, its key value is larger than (or equal to) all those in its left subtree, and smaller than (or equal to) all those in its right subtree.

A Binary Search Tree



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    - A tree with height h can have $2^h - 1$ nodes.
  - We'll study algorithms that maintain search trees in good shape.

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- Pseudocode:

```
INORDER-TREE-WALK(r);
if x ≠ NIL then
    INORDER-TREE-WALK(left(r));
    print key(x);
    INORDER-TREE-WALK(right(r))
end
```

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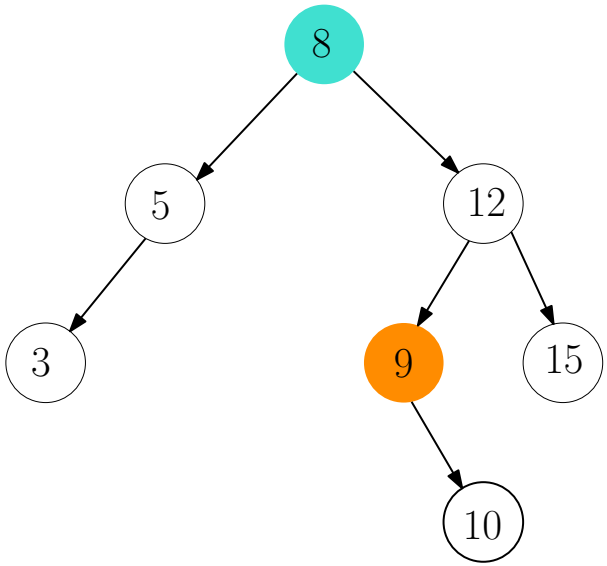
```
Successor(x);
if right(x) ≠ Nil then return Min(right(x)) ;
while parent(x) ≠ Nil do
    y ← parent(x);
    if x = left(y) then return y;
    x ← y;
end
return Nil
```

- Predessor(x): Symmetric to Successor.

Illustration: Successor



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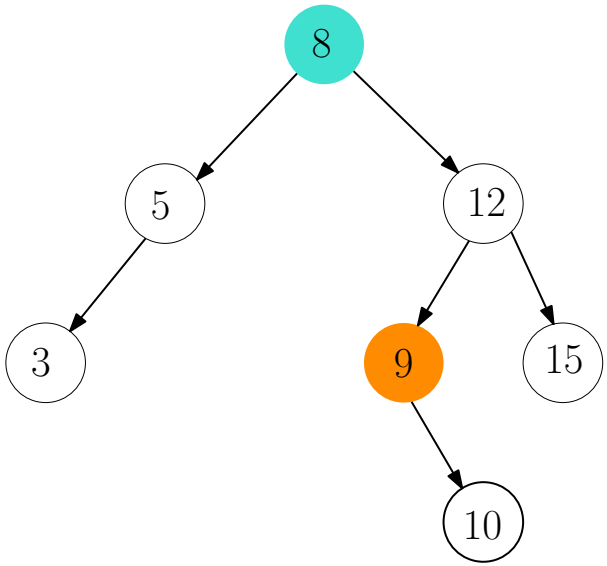

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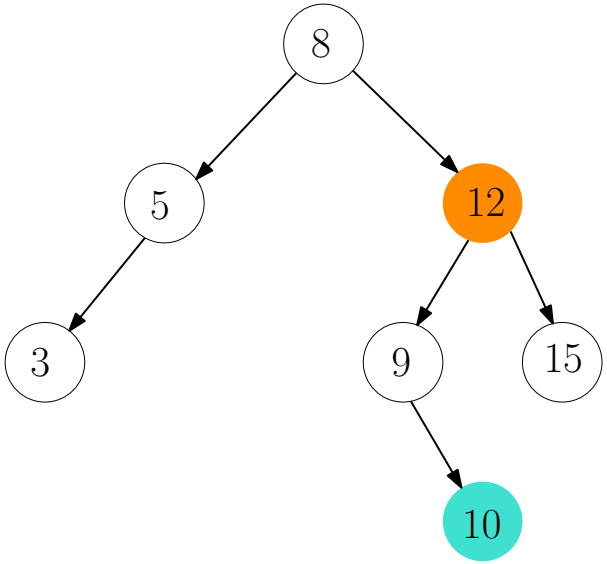

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- Note that every newly inserted node becomes a leaf.

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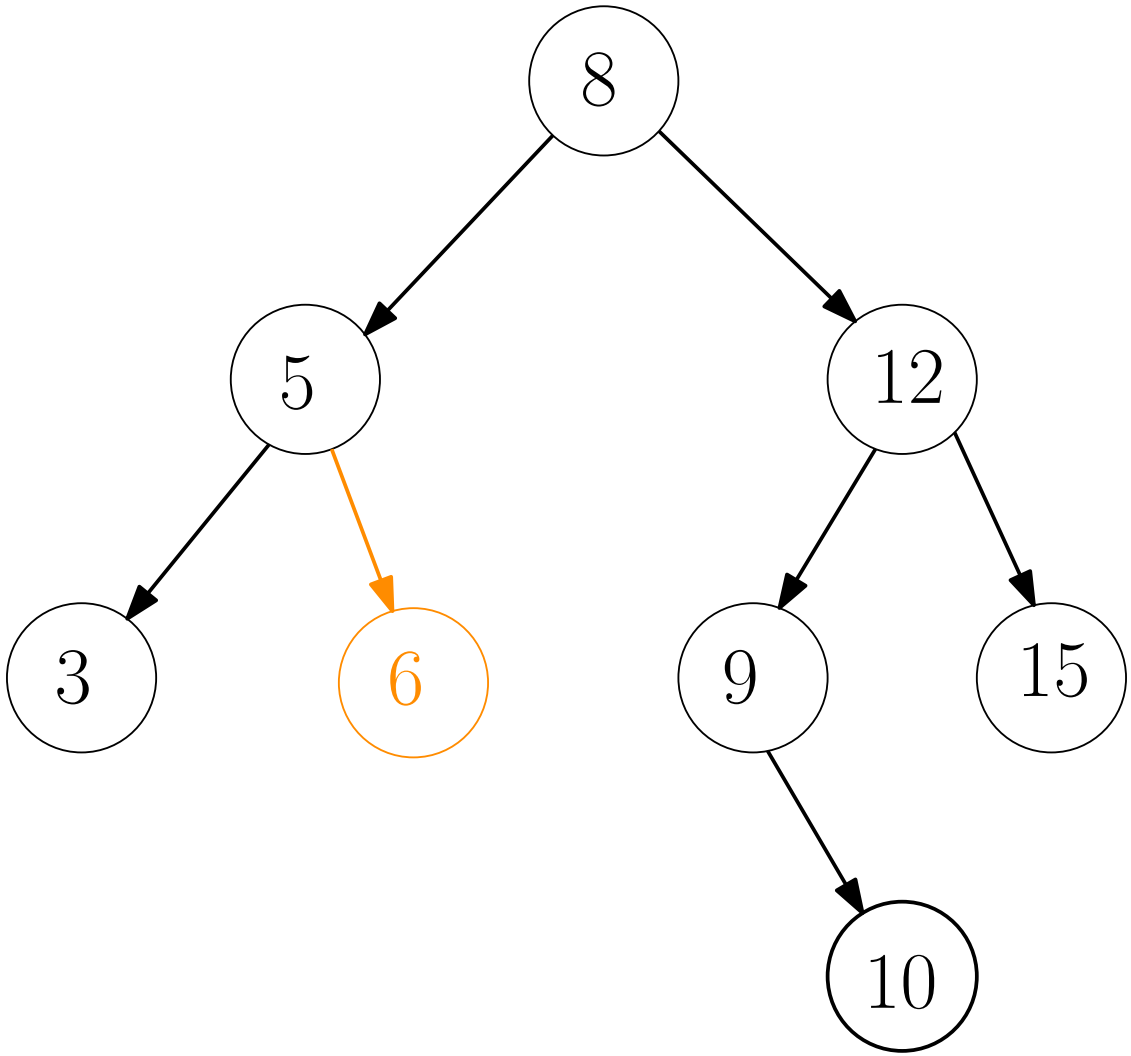

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  - We can replace x by y, and delete y.

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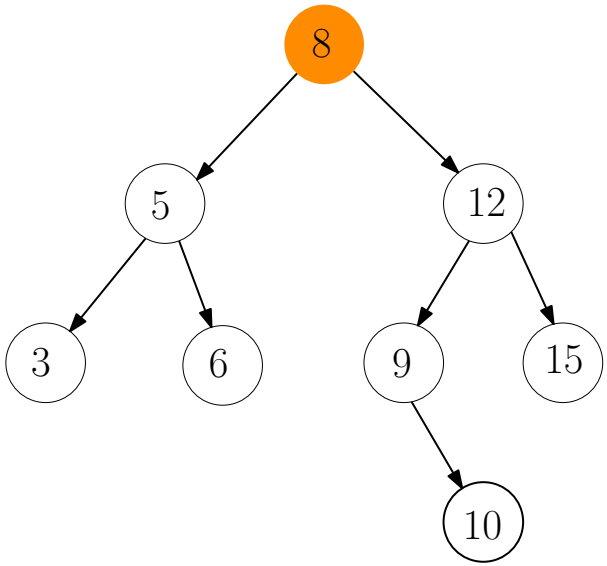

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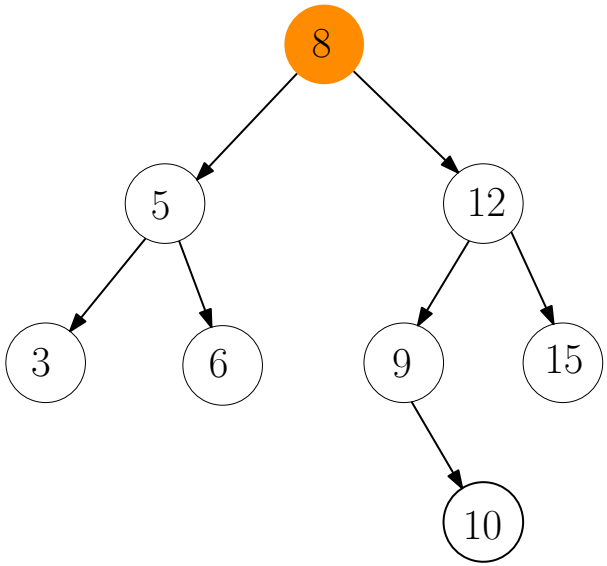

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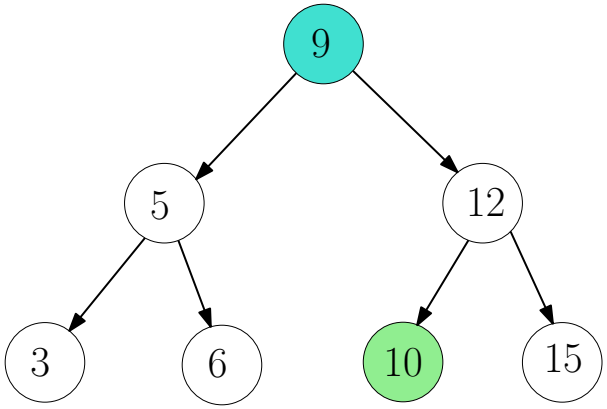

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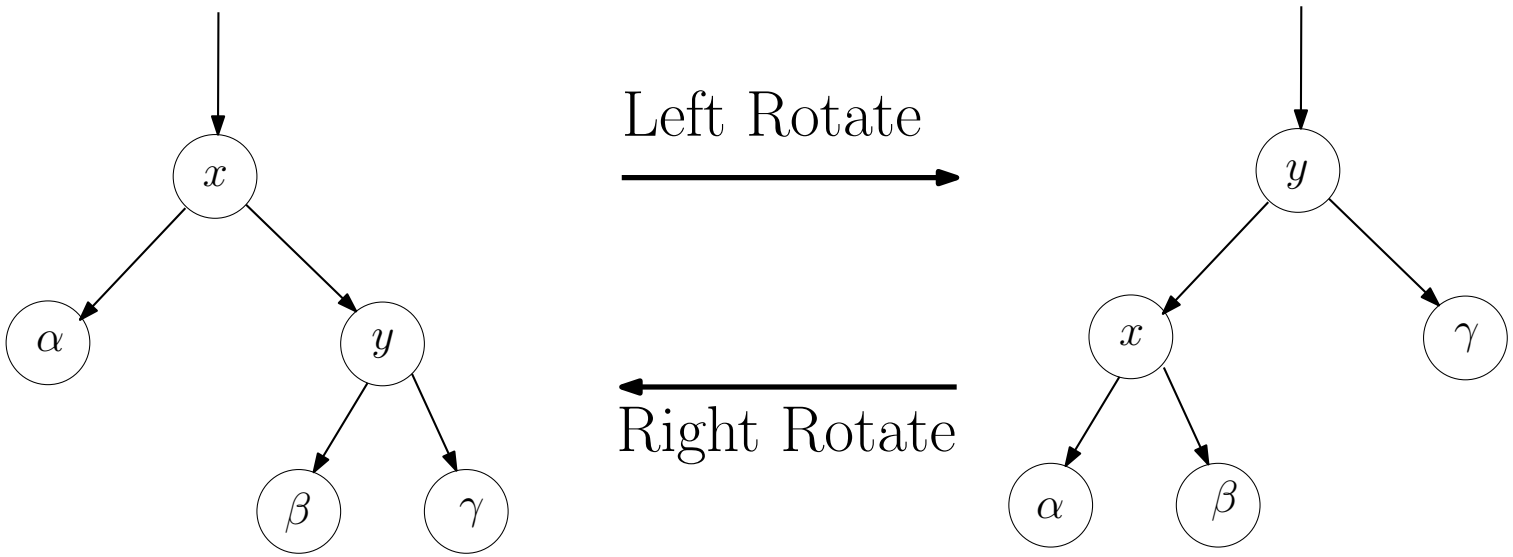

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  - Traversal example: return all entries in increasing key values. Very natural with BST, but inefficient with hashing.